On the effective reflection properties of the randomly segmented elastic bar

Z. KOTULSKI*

ABSTRACT. — In this paper the transition of wave pulses through bars with random properties is investigated. It is assumed that the bar is composed of several homogeneous segments. It is permitted that the lengths of the segments as well as their material parameters are random variables. The overall properties of such bars are studied. The discussion of the results obtained is illustrated with numerical calculations both for periodic and randomly periodic bars.

1. Introduction

Engineering structures performing their functions undergo external or internal excitations which generate vibrations or waves in their elements. Depending on the desired behavior of the structures such phenomena are an intrinsic part of the function of the structure or must be eliminated to avoid damage. The problem of special interest is propagation of wave pulses through elements of structures. Such a phenomenon takes place in a number of working machines or mechanical tools like ramers and drilling rods, where the wave pulse is an intrinsic part of the machines performance. In some other situations (dumpers, absorbers, fixing elements) the wave pulse plays a destructive role. In both cases it is reasonable to design the elements of structure in the optimal way, depending on its function.

As it is well known, the wave pulse is an (elastic) disturbance of the medium, travelling in space, of limited duration and transports energy. The transmission effect of the wave pulse depends not only on the properties of the element of structure but also on the duration and shape of the pulse itself. In the design procedure these two factors must be taken into account.

In this paper we analyze the particular problem of wave pulse propagation in bars where the elements of the structure are mathematically modeled by one-dimensional partial differential equations. Such models have been widely analyzed in the literature, both analytically and numerically (cf. e.g. [Anderson & Lundberg 1984]; [Lundberg et al., 1979]), mostly in the deterministic case. The purpose of this paper is to study such elements where their properties are regarded as random.

* Institute of Fundamental Technological Research, Polish Academy of Sciences, 00-049 Warsaw, Świętokrzyska 21, Poland.
Starting from a randomized model of the bar we try to obtain its overall properties as the element transporting wave pulses and, what it follows, the energy. We use both the analytical and numerical tools in our investigation.

To describe the model of the bar let us assume that it occupies the part of real number axis $x$ starting from $x = 0$ to $x = d$.

The environment is regarded as a pair of semi-infinite bars expanding from minus infinity to $x = 0$ and from $x = d$ to infinity. Moreover, we assume that the investigated bar has some internal structure; it consists of several (e.g. $N$) segments of the lengths $h_j, \sum_{j=1}^{N} h_j = d$, being themselves homogeneous bars. In our model it is permitted that both the length of the segment as well as its material parameters can be random variables. In such a case also the length of the bar $d$ is a random variable.

The problem of the propagation of wave pulses is posed in such a structure. It is assumed that the pulse $f(x, t)$ is coming from the left environment and reaching the front end of the bar $x = 0$ at the instant of time $t = 0$. Then the pulse partially reflects from the interface and partially transmits to the first segment of the bar. Subsequently the transmitted pulse reflects and transmits at all the interfaces of the segments; moreover, we have also reverberation of all the reflected waves on the panels already passed by the wave front. This multiply reflections and transmission process makes the global picture very complicated.

The mathematical analysis of the wave pulse transition is significantly simplified when one transforms in the description of the problem from the space-time domain to the space-spectrum domain, dealing with the Fourier transform of the wave field with respect to temporal variable $t$. Then the exciting wave pulse is $f(x, \omega)$ and the governing equations are the ordinary differential ones. Such an approach was for example effectively utilized in the papers of Lundberg and coauthors (e.g. [Anderson & Lundberg, 1984]; [Lundberg et al., 1979]); it is also efficient in multi-dimensional problem, e.g. wave propagation in layered media (e.g. [Kennett, 1981]).

The following section is devoted to the formulation of the problem of wave pulse propagation through the segmented bar with the use of the spectral method. Then we consider the periodic case, where the bar is built of a series of identical couples of segments, and the stochastic model. In both cases we obtain the overall reflection properties of the bar, in stochastic case using the law of large numbers for the product of random matrices. In the last section we illustrate the analytical results with the numerical example.

2. Wave pulses in the segmented elastic bar

Wave propagating along the length of an elastic bar of a constant cross-section is described by the system of two differential equations (see [Lundberg et al., 1979]):
\[
\begin{aligned}
\frac{\partial f}{\partial x} &= A \rho \frac{\partial v}{\partial t} \\
\frac{\partial v}{\partial x} &= \frac{1}{AE} \frac{\partial f}{\partial t}
\end{aligned}
\]

where

\( f \) denotes the force,
\( v \) is the particle velocity in the medium

and

\( A \) is the area of the perpendicular cross-section of the bar,
\( \rho \) is the density of the material,
\( E \) is the Young modulus,

and \( x, t \) are respectively, the spatial variable along the length of the bar and time.

Introducing as a new parameter the characteristic impedance \( Z = A \sqrt{\rho E} \), and, instead of the spatial variable \( x \), a new independent variable being the wave travel time from 0 to \( x, \xi = x/c \), where \( c = \sqrt{\frac{E}{\rho}} \) is the velocity, we can write the equation (2.1) in the following form:

\[
\frac{\partial}{\partial \xi} s = Q \frac{\partial}{\partial t} s,
\]

where

\[
Q = \begin{bmatrix} 0 & Z \\ 1 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} f \\ v \end{bmatrix}.
\]

Finally, we apply in Eq. (2.2) the Fourier transformation with respect to time \( t \):

\[
\hat{s}(\xi, \omega) = \int_{-\infty}^{\infty} e^{-i\omega t} s(\xi, t) dt, \quad s(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{s}(\xi, \omega) d\omega,
\]

and obtain the ordinary differential equation for the transformed wave field:

\[
\frac{d}{d\xi} \hat{s} = i\omega Q \hat{s},
\]

with some initial condition \( \hat{s}(0, \omega) \). Its solution in an uniform bar can be represented in the following form:

\[
\hat{s}(\xi, \omega) = P(\xi, \omega) \hat{s}(0, \omega),
\]

where \( P(\xi, \omega) \) is the solution of the following matrix differential equation:
\[ \frac{d}{d\xi} P = i\omega QP, \quad P(0, \omega) = \text{Id}. \]

Such a solution has the following form:

\[ P = e^{i\omega Q\xi} = \frac{1}{2} \begin{bmatrix} e^{i\omega\xi} + e^{-i\omega\xi} & Z \left( -e^{i\omega\xi} + e^{-i\omega\xi} \right) \\ \frac{1}{Z} \left( -e^{i\omega\xi} + e^{-i\omega\xi} \right) & e^{i\omega\xi} + e^{-i\omega\xi} \end{bmatrix}. \]

This idealized situation complicates when the travelling wave pulse reaches a surface of discontinuity where some material parameters of the bar changes. Then the wave is partially reflected and partially transmitted. At the two sides of the interface of two homogeneous panels of the bar the amplitude of the wave and its phase jump. However, the values of these parameters at both sides of the interface point \( \xi \) are connected by the continuity condition of the force and velocity field:

\[ \hat{s}(\xi^-, \omega) = \hat{s}(\xi^+, \omega) \]

Restricting our interest to the force only we can eliminate the velocity field \( \hat{v} \) from the Eq. (2.5). The resulting equation for \( \hat{f} \) is:

\[ \frac{d^2}{d\xi^2} \hat{f} + \omega^2 \hat{f} = 0 \]

which has the solution

\[ \hat{f}(\xi, \omega) = \hat{f}_R(\omega) e^{-i\omega\xi} + \hat{f}_L(\omega) e^{i\omega\xi} \]

where

\[ \hat{f}_R(\omega) \] is the amplitude of the right-going (incident) wave,
\[ \hat{f}_L(\omega) \] is the amplitude of the left-going (reflected) wave.

We can express the velocity field \( \hat{v}(\xi, \omega) \) in terms of the amplitudes \( \hat{f}_R(\omega) \) and \( \hat{f}_L(\omega) \) (see [Lundberg et al., 1979]) as:

\[ \hat{v}(\xi, \omega) = \frac{1}{Z} \left( -\hat{f}_R(\omega) e^{-i\omega\xi} + \hat{f}_L(\omega) e^{i\omega\xi} \right). \]

Then the continuity condition (2.9) can also be expressed in terms of the amplitudes of the left- and right-going waves.

Now assume that the bar is built of \( N \) homogeneous panels; in the \( j \)-th panel the characteristic impedance is \( Z_j \), and the wave travel time is \( h_j \). The beginning of the bar is located at the point 0; the following points of interface of the panels are in the travel time domain \( \xi_j = \sum_{k=1}^{j} h_k \). Assume also that the wave pulse \( \hat{f}_R^0 e^{-i\omega\xi} \) comes from
the surrounding media (with the impedance indexed by 0) to the front end of the bar. Then the reflected pulse is generated in media 0 and a transmitted wave in panel 1. Subsequently due to the sequence of interfaces of the panels (discontinuity points) we have right and left-going waves of the following form:

\[(2.13) \quad \Phi_R (\omega, \xi) = \hat{f}_R^j e^{-i\omega \xi}, \quad \Phi_L (\omega, \xi) = \hat{f}_L^j e^{i\omega \xi},\]

in the panel with the travel time \(h_j\), i.e. for \(\xi \in (\xi_{j-1}, \xi_j)\). In the surrounding media behind the bar there is a right-going (transmitted wave) of the form:

\[(2.14) \quad \Phi_R (\omega, \xi) = \hat{f}_R^{N+1} e^{i\omega \xi}.\]

This means that the continuity conditions at the interfaces of the segments, written in matrix form, for the particular point \(\xi_{j-1}\) are:

\[(2.15) \quad \begin{bmatrix} 1 & 1 \\ -\frac{1}{Z_{j-1}} & \frac{1}{Z_{j-1}} \end{bmatrix} \begin{bmatrix} \hat{f}_R^{j-1} e^{-i\omega \xi_{j-1}} \\ \hat{f}_L^{j-1} e^{i\omega \xi_{j-1}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{Z_j} & \frac{1}{Z_j} \end{bmatrix} \begin{bmatrix} \hat{f}_R^j e^{-i\omega \xi_j} \\ \hat{f}_L^j e^{i\omega \xi_j} \end{bmatrix}.\]

Let us introduce new variables:

\[(2.16) \quad \varphi_R^j = \hat{f}_R^j e^{-i\omega \xi_j}, \quad \varphi_L^j = \hat{f}_L^j e^{i\omega \xi_j}.\]

Since \(\xi_{j-1} = \xi_j - h_j\) (or \(\xi_j = \xi_{j-1} + h_j\)) for \(j = 1, 2, \cdots, N\); \(\xi_0 = 0, h_0 = 0\), we obtain:

\[(2.17) \quad \hat{f}_R^{j-1} e^{-i\omega \xi_{j-1}} = \hat{f}_R^j e^{-i\omega \xi_{j-2}} e^{-i\omega h_{j-1}} = \varphi_R^{j-1} e^{-i\omega h_{j-1}},\]

\[(2.18) \quad \hat{f}_L^{j-1} e^{i\omega \xi_{j-1}} = \hat{f}_L^j e^{i\omega \xi_{j-2}} e^{i\omega h_{j-1}} = \varphi_L^{j-1} e^{i\omega h_{j-1}},\]

and the continuity equation takes the form:

\[(2.19) \quad \begin{bmatrix} 1 & 1 \\ -\frac{1}{Z_{j-1}} & \frac{1}{Z_{j-1}} \end{bmatrix} \begin{bmatrix} e^{-i\omega h_{j-1}} \\ 0 \end{bmatrix} \begin{bmatrix} \varphi_R^{j-1} \\ \varphi_L^{j-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{Z_j} & \frac{1}{Z_j} \end{bmatrix} \begin{bmatrix} \varphi_R^j \\ \varphi_L^j \end{bmatrix}.\]

Solving this equation with respect to \((j-1)\)-th variable and introducing the new notation:

\[(2.20) \quad A_j = \begin{bmatrix} 1 & 1 \\ -\frac{1}{Z_j} & \frac{1}{Z_j} \end{bmatrix}, \quad E_j = \begin{bmatrix} e^{i\omega h_{j-1}} \\ 0 \end{bmatrix}, \quad \hat{\varphi}_j^0 = \begin{bmatrix} \varphi_R^j \\ \varphi_L^j \end{bmatrix},\]

we obtain the following transition equation:
\( \hat{F}^{j-1} = E_{j-1} A_{j-1}^{-1} A_j \hat{F}^j. \)

Consider the bar consisting of \( N \) elements. We have (\( \hat{F}^0 \) corresponds to the left environment, \( \hat{F}^{N+1} \) to the right environment):

\( \hat{F}^0 = E_0 A_0^{-1} A_1 E_1 A_1^{-1} \cdots A_N E_N A_N^{-1} A_{N+1} \hat{F}^{N+1}. \)

Since \( E_0 \equiv \text{Id} \), this equation can be written in the following form:

\( \hat{F}^0 = A_0^{-1} \prod_{j=1}^{N} A_j E_j A_j^{-1} \cdot A_{N+1} \hat{F}^{N+1}. \)

We can write the transition matrix \( A_j E_j A_j^{-1} \) explicitly. Since:

\( A_j^{-1} = \begin{bmatrix} \frac{1}{2} & -Z_j \frac{1}{2} \\ \frac{1}{2} & \frac{Z_j}{2} \end{bmatrix}, \)

we obtain:

\( A_j E_j A_j^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\omega h_{j-1}} & 0 \\ 0 & e^{-i\omega h_{j-1}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{Z_{j-1}}{2} \\ \frac{1}{2} & \frac{Z_{j-1}}{2} \end{bmatrix} \)

\( = \begin{bmatrix} \cos \omega h_j & \frac{1}{Z_j} \sin \omega h_j \\ -i \frac{1}{Z_j} \sin \omega h_j & \cos \omega h_j \end{bmatrix}. \)

The transition matrix \( A_j E_j A_j^{-1} \) also has a real representation; it can be obtained by multiplying the matrix by matrix \( R \) and its inverse defined as:

\( R = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}. \)

Then the transition matrix is:

\( M_j = R^{-1} A_j E_j A_j^{-1} R = \begin{bmatrix} \cos \omega h_j & Z_j \sin \omega h_j \\ -\frac{1}{Z_j} \sin \omega h_j & \cos \omega h_j \end{bmatrix} \)

and the equation for the entire bar becomes:

\( \hat{F}^0 = A_0^{-1} R \prod_{j=1}^{N} M_j R^{-1} \cdot A_{N+1} \hat{F}^{N+1}. \)
Returning to the original amplitudes \( f^0_R, f^0_L, f^{N+1}_R \) (since in the right half-space there is no reflected wave, we have \( f^{N+1}_L \equiv 0 \)), we have:

\[
\begin{align*}
\varphi^0_R &= f^0_R, \\
\varphi^0_L &= f^0_L, \\
\varphi^{N+1}_R &= f^{N+1}_R e^{-i\omega \xi_R} = f^{N+1}_R \exp \left\{ -i\omega \sum_{j=1}^{N} h_j \right\},
\end{align*}
\]

(2.29)

and the wave Eq. (2.28) becomes:

\[
\begin{align*}
\begin{bmatrix}
\dot{f}^0_R \\
\dot{f}^0_L
\end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & -iZ_0 \\
1 & iZ_0
\end{bmatrix} \prod_{j=1}^{N} M_j \begin{bmatrix} 1 \\
\frac{i}{Z_{N+1}}
\end{bmatrix} \begin{align*}
\dot{f}^{N+1}_R \exp \left\{ -i\omega \sum_{j=1}^{N} h_j \right\}.
\end{align*}
\end{align*}
\]

(2.30)

3. The model of the periodic bar

Consider now a periodic case, which is such that the bar consists of two kinds of elements, the couples of which are located periodically. Let \( N = 2K \) and the transition matrices through the layers (of the form defined in (2.27)) are:

\[
\begin{align*}
M_i &= M_1 \quad \text{for} \quad i = 1, 3, \ldots, 2K - 1, \\
M_i &= M_2 \quad \text{for} \quad i = 2, 4, \ldots, 2K.
\end{align*}
\]

The travel times \( h_1 \) and \( h_2 \) are related to the lengths of the panels \( l_1 \) and \( l_2 \) according to the formula:

\[
h_i = \frac{l_i}{c_i} = \sqrt{\frac{\rho_i}{E_i}} l_i = \frac{A \rho_i}{Z_i} l_i, \quad i = 1, 2.
\]

(3.1)

We can introduce the transfer matrix through the couple of layers:

\[
M = M_1 M_2
\]

(3.2)

and transform the equation for the amplitudes to the following form:

\[
\begin{align*}
\begin{bmatrix}
\dot{f}^n_R \\
\dot{f}^n_L
\end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & -iZ_0 \\
1 & iZ_0
\end{bmatrix} M^K \begin{bmatrix} 1 \\
\frac{i}{Z_{hn}}
\end{bmatrix} \begin{align*}
f^{hn}_R \exp \left\{ -i\omega K(h_1 + h_2) \right\}
\end{align*}
\end{align*}
\]

(3.3)
Consider now a limit case where the number of panels in the bar tends to infinity but their lengths tend to zero in such a way that the length of the bar as well as the ratio of the lengths of its components remain constant:

\[(3.4) \quad K \to \infty, \quad l_1 = \frac{L_1}{K}, \quad l_2 = \frac{L_2}{K} \]

and it follows:

\[(3.5) \quad h_1 = \frac{H_1}{K}, \quad h_2 = \frac{H_2}{K} \]

where

\[H_1 = \frac{A \rho_1}{Z_1} L_1, \quad H_2 = \frac{A \rho_2}{Z_2} L_2,\]

are the travel time periods through all the segments within the bar made, respectively, of material 1 and 2.

It can be shown that in the limit the reflection properties of the bar are equivalent to the properties of some uniform homogenized bar with effective parameters (the characteristic impedance and the travel time from the beginning of the bar to its end). Their numerical values can be calculated analogously to the effective properties of the stratified slab obtained in [Kotulski, 1990, 1992]. To calculate them let us substitute the assumed values of \( h_1 = \frac{H_1}{K} \) and \( h_2 = \frac{H_2}{K} \) into the expression (3.2) for the transition matrix \( M \).

Expanding the matrix \( M \) in a power series we have:

\[(3.7) \quad M = \text{Id} + \frac{1}{K} B + o \left( \frac{1}{K} \right),\]

where \( \text{Id} \) is a \( 2 \times 2 \) identity matrix and

\[(3.8) \quad B = \begin{bmatrix} \omega H_1 & Z_1 \omega H_1 + Z_2 \omega H_2 \\ -\frac{Z_1 \omega H_1}{Z_2} & \omega H_2 \end{bmatrix}.\]

Calculating the limit

\[(3.9) \quad \lim_{K \to \infty} M^K = \lim_{K \to \infty} \left( \text{Id} + \frac{1}{K} B + o \left( \frac{1}{K} \right) \right)^K = e^B,\]

we obtain the effective transition matrix for the homogenized bar:

\[(3.10) \quad e^B = \frac{1}{2} \begin{bmatrix} e^{-i\omega a} + e^{i\omega a} & \frac{i b}{1} (e^{-i\omega a} - e^{i\omega a}) \\ \frac{1}{i b} (e^{-i\omega a} - e^{i\omega a}) & e^{-i\omega a} + e^{i\omega a} \end{bmatrix} = \begin{bmatrix} \cos \omega a & b \sin \omega a \\ -\frac{1}{b} \sin \omega a & \cos \omega a \end{bmatrix}.\]
where the effective travel time through the bar $a$ and the effective characteristic impedance $b$ are equal to:

\begin{equation}
(3.11) \quad a = \sqrt{\frac{(H_1 Z_1 + H_2 Z_2)(H_1 Z_2 + H_2 Z_1)}{Z_1 Z_2}},
\end{equation}

\begin{equation}
(3.12) \quad b = \sqrt{Z_1 Z_2 \frac{H_1 Z_1 + H_2 Z_2}{H_1 Z_2 + H_2 Z_1}}.
\end{equation}

The effective parameters $a$ and $b$ can also be expressed in terms of the lengths $L_1$ and $L_2$ and the material parameters, instead of the travel times and characteristic impedances.

The above formulae prove that the effective parameters of the material in the considered dynamic (time-dependent) model are the same as in static or harmonic case (see [Kotulski, 1992]).

Substituting the effective transfer matrix into the equation for the amplitudes of the waves (3.3) we obtain following final equation:

\begin{equation}
(3.13) \quad \begin{bmatrix} f_R^0 \\ f_L^0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i Z_0 \\ 1 & i Z_0 \end{bmatrix} \begin{bmatrix} \cos \omega a & b \sin \omega a \\ -\frac{1}{b} \sin \omega a & \cos \omega a \end{bmatrix} \begin{bmatrix} 1 \\ i \frac{Z_{\text{fin}}}{Z_{\text{fin}}} \end{bmatrix} f_{\text{fin}} R e^{-i\omega(H_1 + H_2)}.
\end{equation}

Multiplying the matrices in (3.13) we obtain the equation for the amplitudes in an explicit form:

\begin{equation}
(3.14) \quad f_R^0 (\omega) = \frac{1}{2} \left[ \left(1 + \frac{Z_0}{Z_{\text{fin}}} \right) \cos \omega a + i \left(\frac{b}{Z_{\text{fin}}} + \frac{Z_0}{b}\right) \sin \omega a \right] f_{\text{fin}} R (\omega) e^{-i\omega(H_1 + H_2)},
\end{equation}

\begin{equation}
(3.15) \quad f_L^0 (\omega) = \frac{1}{2} \left[ \left(1 - \frac{Z_0}{Z_{\text{fin}}} \right) \cos \omega a + i \left(\frac{b}{Z_{\text{fin}}} + \frac{Z_0}{b}\right) \sin \omega a \right] f_{\text{fin}} R (\omega) e^{-i\omega(H_1 + H_2)}.
\end{equation}

In posing the problem we have assumed that $f_R^0 (\omega)$, the amplitude of the incident pulse, is the known quantity in our model. Therefore we can express the required amplitude of the reflected wave with the use of $f_R^0 (\omega)$, in the following form:

\begin{equation}
(3.16) \quad f_L^0 (\omega) = \frac{\left[ \left(1 - \frac{Z_0}{Z_{\text{fin}}} \right) \cos \omega a + i \left(\frac{b}{Z_{\text{fin}}} - \frac{Z_0}{b}\right) \sin \omega a \right] f_R^0 (\omega)}{\left[ \left(1 + \frac{Z_0}{Z_{\text{fin}}} \right) \cos \omega a + i \left(\frac{b}{Z_{\text{fin}}} + \frac{Z_0}{b}\right) \sin \omega a \right]}.
\end{equation}

Calculating the inverse Fourier transform of the expression (3.16) we obtain the behavior (the shape) of the reflected pulses in time.
4. The model of the bar with random properties

In the model of the bar considered in the previous section it is assumed that both the material parameters and the geometrical dimensions of the bar are deterministic. In reality, since the element of the structure is built in factory conditions (some tolerances in dimensions, the pieces of the material selected from some bigger sample, etc.), such quantities should be regarded as random variables. Then, the waves generated by the incident deterministic wave pulse prove to have stochastic properties and must be regarded as stochastic processes.

In practice we are mostly interested in some averaged properties of such transition phenomena – average wave amplitudes, average transmitted (reflected) energy and – in some limit case – the overall properties of the bar. In this section we apply the law of large numbers for the product of random matrices (cf. [Berger 1984]) to obtain the effective transmission properties of a bar built of a large number of segments with random properties.

The Eq. (2.30) satisfied by the amplitudes of the reflected and transmitted waves are valid also in the case when the material parameters and the lengths of the segments are random variables. Therefore, for every finite number of elements in the bar the equations so obtained (with an appropriate stochastic interpretation) can describe the wave field. The situation become more complicated in the limit case when the number of segments tends to infinity. However, the law of large numbers ensures that the problem can be successfully solved.

Assume that the bar is built of \( 2K \) segments with lengths \( l_1(\gamma), l_2(\gamma), \ldots, l_{2K}(\gamma) \), where \( l_i(\gamma), i = 1, 2, \ldots, 2K \) are random variables. In the above \( \gamma \in \Gamma \) is an elementary event and \( (\Gamma, \mathcal{F}, \mathcal{P}) \) is the complete probabilistic space (cf. [Sobczyk, 1991]). Assume additionally that the material parameters of the segments and the areas of their cross-sections \( (\rho_{2j-1}(\gamma), E_{2j-1}(\gamma), A_{2j-1}(\gamma), \rho_{2j}(\gamma), E_{2j}(\gamma), A_{2j}(\gamma)) \) are, as the vector random variables, independent and identically distributed for \( j = 1, 2, \ldots, 2K \). Moreover, we assume that the lengths of the segments have the following particular property:

\[
(4.1) \quad (l_{2j-1}(\gamma), l_{2j}(\gamma)) = \left( \frac{l_{2j-1}(\gamma)}{2K}, \frac{l_{2j}(\gamma)}{2K} \right),
\]

for \( j = 1, 2, \ldots, K \) are independent, identically distributed two-dimensional random variables with

\[
(4.2) \quad E\{l_{2j-1}(\gamma)\} = L_1, \quad E\{l_{2j}(\gamma)\} = L_2,
\]

for \( j = 1, 2, \ldots, K \). In this particular case the Eq. (2.30) for the Fourier transform of the amplitudes takes the following form:

\[
(4.3) \quad \begin{cases}
\begin{bmatrix}
\mathcal{F}^R_j(\omega, \gamma) \\
\mathcal{F}^L_j(\omega, \gamma)
\end{bmatrix}
= \frac{1}{2} \begin{bmatrix}
1 & -i Z_0 \\
1 & i Z_0
\end{bmatrix} \prod_{j=1}^{K} M_j(\omega, \gamma) \\
\times \begin{bmatrix}
1 \\
\frac{i}{Z_{2K+1}}
\end{bmatrix} \mathcal{F}^{2K+1}_R(\omega, \gamma) \exp \left\{ -i \omega \sum_{j=1}^{2K+1} h_j \right\}
\end{cases}
\]

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where \( h_j(\gamma) \) are the randomized counterparts of the travel time, \( M_j(\omega, \gamma) \) are the randomized transfer matrices through the couples of segments defined in (3.2):

\[
(4.4) \quad M_j(\omega, \gamma) = \begin{bmatrix} \cos \omega h_{2j-1}(\gamma) \cos \omega h_{2j}(\gamma) - \frac{Z_{2j-1}(\gamma)}{Z_{2j}(\gamma)} \sin \omega h_{2j-1}(\gamma) \sin \omega h_{2j}(\gamma), \\
- \frac{\sin \omega h_{2j-1}(\gamma) \cos \omega h_{2j}(\gamma) - \frac{\cos \omega h_{2j-1}(\gamma) \sin \omega h_{2j}(\gamma)}{Z_{2j}(\gamma)}}{Z_{2j-1}(\gamma) \sin \omega h_{2j-1}(\gamma) \cos \omega h_{2j}(\gamma) + Z_{2j}(\gamma) \cos \omega h_{2j-1}(\gamma) \sin \omega h_{2j}(\gamma)}, \\
\cos \omega h_{2j-1}(\gamma) \cos \omega h_{2j}(\gamma) - \frac{Z_{2j}(\gamma)}{Z_{2j-1}(\gamma)} \sin \omega h_{2j-1}(\gamma) \sin \omega h_{2j}(\gamma) \end{bmatrix},
\]

for \( j = 1, 2, \ldots, K, \) with

\[
(4.5) \quad Z_i(\gamma) = A_i(\gamma) \sqrt{\rho_i(\gamma)} E_i(\gamma), \quad i = 1, 2, \ldots, 2K.
\]

To study the asymptotic behavior of the randomized equation for the amplitudes of the waves we apply the law of large numbers for the products of random matrices obtained in [Berger, 1984]. This theorem can be written in the following form.

Consider the sequence of the products of real random matrices

\[
(4.6) \quad P_K(\gamma) = \prod_{j=1}^{K} M_{j,K}(\gamma).
\]

It is assumed that for \( K \) tending to infinity the matrices \( M_{j,K} \) can be represented as

\[
(4.7) \quad M_{j,K}(\gamma) = \text{Id} + \frac{1}{K} B_{j,K}(\gamma) + R_{j,K}(\gamma),
\]

where \( B_{j,K}(\gamma) \) for \( j = 1, 2, \ldots, K \) are independent, identically distributed random matrices, integrable with respect to probability measure \( P \) and \( |R_{j,K}(\gamma)| = o(K^{-1}) \) for large \( K \). Under these conditions the law of large numbers takes place and

\[
(4.8) \quad \lim_{K \to \infty} P_K(\gamma) = \exp \{ E \{ B_{j,K}(\gamma) \} \},
\]

in the sense of convergence in distribution of all the vectors obtained from multiplication of the random matrix by an arbitrary deterministic vector.

To analyse the limit case of the propagation of wave pulses through a bar built of segments we decompose the transition matrix defined in (4.4) under the assumption (4.1) \( (h_j(\gamma) \) are connected with \( l_j(\gamma) \) by the formula analogous to (3.6)) with respect to the powers of \( 1/K \), where the matrices \( B_j \) required in formula (4.8) are defined as:

\[
(4.9) \quad B_j = \begin{bmatrix} 0 & \frac{Z_{2j-1}(\gamma) \omega H_{2j-1}(\gamma) + Z_{2j}(\gamma) \omega H_{2j}(\gamma)}{Z_{2j}(\gamma)} \\
- \frac{\omega H_{2j-1}(\gamma)}{Z_{2j-1}(\gamma)} & 0 \end{bmatrix},
\]
and their common average value is

\begin{equation}
E \{ B_j \} = \omega \left[ -E \left\{ \frac{H_1(\gamma)}{Z_1(\gamma)} \right\} - E \left\{ \frac{H_2(\gamma)}{Z_2(\gamma)} \right\} \right. \\
\left. E \{ Z_1(\gamma) H_1(\gamma) \} + E \{ Z_2(\gamma) H_2(\gamma) \} \right].
\end{equation}

(4.10)

The matrix \( e^{E \{ B_j \} } \) is of the form analogous to (3.10) where at present the effective travel time through the bar \( a \) and the effective characteristic impedance \( b \) are equal to:

\begin{equation}
a = \sqrt{\left( E \{ Z_1(\gamma) H_1(\gamma) \} + E \{ Z_2(\gamma) H_2(\gamma) \} \right) \left( E \left\{ \frac{H_1(\gamma)}{Z_1(\gamma)} \right\} + E \left\{ \frac{H_2(\gamma)}{Z_2(\gamma)} \right\} \right)},
\end{equation}

(4.11)

\begin{equation}
b = \sqrt{\frac{E \{ Z_1(\gamma) H_1(\gamma) \} + E \{ Z_2(\gamma) H_2(\gamma) \}}{E \left\{ \frac{H_1(\gamma)}{Z_1(\gamma)} \right\} + E \left\{ \frac{H_2(\gamma)}{Z_2(\gamma)} \right\}}.
\end{equation}

(4.12)

The above formulae can be easily generalized to the case where the bar is built of more than two kinds of material – the period of the segments (in our stochastic sense) is e.g. \( k \). Then the transfer matrix for the homogenized bar is also of the form (3.10) with the effective travel time \( a \) and the effective characteristic impedance \( b \) are defined as

\begin{equation}
a = \sqrt{\sum_{i=1}^{k} E \{ \rho_i(\gamma) A_i(\gamma) L_i(\gamma) \} \sum_{i=1}^{k} E \left\{ \frac{L_i(\gamma)}{E_i(\gamma) A_i(\gamma)} \right\}},
\end{equation}

(4.13)

\begin{equation}
b = \sqrt{\left( \sum_{i=1}^{k} E \{ \rho_i(\gamma) A_i(\gamma) L_i(\gamma) \} \right) \left( \sum_{i=1}^{k} E \left\{ \frac{L_i(\gamma)}{E_i(\gamma) A_i(\gamma)} \right\} \right)}.
\end{equation}

(4.14)

5. Illustrative example and discussion

In the considerations of this section let us concentrate on the reflected pulses characterized for the homogenized (uniform) bar by formula (3.16) and for the segmented bar by an analogous formula obtained directly from Eq. (3.3):

\begin{equation}
\tilde{f}_L(\omega) = \frac{\left( M_{11}^K - M_{22}^K \frac{Z_0}{Z_{fin}} \right) + i \left( M_{12}^K \frac{1}{Z_{fin}} + M_{21}^K Z_0 \right)}{\left( M_{11}^K + M_{22}^K \frac{Z_0}{Z_{fin}} \right) + i \left( M_{12}^K \frac{1}{Z_{fin}} - M_{21}^K Z_0 \right)} \int_R \omega,
\end{equation}

(5.1)
where $M_{ij}^K$ is the $ij$-th element of the matrix $M^K$. Calculating the inverse Fourier transform of the above expression we obtain the shape of the reflected pulse in the temporal domain and, taking its absolute value, the changes in time of the amplitude of the reflected wave.

As it is seen from the formula (3.16) or (5.1), the reflected pulse is the function of two components: the initial pulse, characterized by $f_R^0(\omega)$ and the material properties of the dynamic system, characterized by the remaining part of the formulae. The purpose of this section is the numerical studying of the convergence of the solution for segmented bar to the homogenized case when the number of segments tends to infinity. This fact determines the material (geometrical) part of the equation for the reflected pulse. The shape of the initial pulse $f_R^0(\omega)$, in this particular problem plays the role of a testing tool and may be arbitrarily chosen. Therefore we can select such a form of the initial pulse which gives us the possibility of inspecting with the best accuracy what happens inside the reflecting bar. The rectangular pulse seems to have such a property – it starts rapidly and has finite duration. This means that we know when the pulse reflected at a given interface starts and how long it continues.

Writing this in terms of the formulae, we assume that the initial rectangular pulse has the following form:

\begin{equation}
(5.2) \quad f(t) = \begin{cases} 
\beta, & t \in (0, \alpha) \\
0, & \text{otherwise}
\end{cases},
\end{equation}

where $\beta$ is the value of the amplitude of the pulse (the displacement of the material) and $\alpha$ is the duration of the pulse (starting at time $t = 0$). Then the Fourier transform of the pulse required in the formula (5.1) is the following function of the spectral parameter $\omega \in (-\infty, \infty)$:

\begin{equation}
(5.3) \quad \tilde{f}_R(\omega) = \alpha \beta \left( \sin \frac{\omega \alpha}{2} \right) \exp \left\{ -i \frac{\omega \alpha}{2} \right\}.
\end{equation}

The fact that the rectangular initial pulse gives a very distinct shape of the reflected pulse is very important since even in the case of homogeneous reflecting bar its shape can be quite complicated. It can be observed that the ratio of the duration of the pulse and the travel time through the bar has the strong effect on the shape of the reflected pulse.

The much more complicated situation is when the reflecting bar consists of several segments. In this example we wish to present how the homogenization procedure, theoretically analyzed in the previous sections, works in the case of a concrete bar.

Assume that our periodic bar analyzed in Section 3 is built of two kinds of material with characteristic impedances, respectively, $Z_1 = 4.0$ and $Z_2 = 2.0$. We assume that the surrounding medium has an impedance equal to 1 (that is, in our formula $Z_0 = Z_{\text{fin}} = 1.0$). Moreover, we assume that the length of the bar is finite and both materials participate in it in such a way, that the travel time through both of them is the same, equal to 1 (in (3.5-3.8) $H_1 = H_2 = 1.0$). For such parameters in our periodic
model analyzed in Section 3, we can calculate the numerical values of the effective travel time and the effective characteristic impedance of the homogenized medium according to the formulae (3.11)-(3.12). The result is

\[(5.4) \quad a = 2.12132, \quad b = 2.828427.\]

It is seen that the effective travel time is higher than the sum of the travel times through its components. This means that the wave pulse localizes for a certain time in the segmented bar.

To know something more about the reflection process in the case when the number of segments in the bar is finite we must perform numerical calculations. In particular, to find the amplitude of the reflected pulse, we must calculate the inverse Fourier transform of the expression defined in (5.1), which in the numerical case is the discrete Fourier transform (see [Doyle, 1989]). The algorithm for the numerical calculation of the amplitude of the reflected pulse requires very careful application of the numerical procedures.

The inverse Fourier transform of the function \(F(\omega)\) (in our example \(F(\omega) = F^R(\omega)\), \(\omega \in (-\infty, \infty)\) is a spectral parameter), defined in (2.4), can be written in the discrete form as:

\[(5.5) \quad f(t) = \frac{1}{2\pi} \sum (TD_n) e^{i\omega t} \Delta \omega\]

where

\[(5.6) \quad D_n = F(\omega_n), \quad \omega_n = n \Delta \omega, \quad \Delta \omega = \frac{2\pi}{T},\]

and \(T\) is a constant regarded as a period of the temporal function \(f(t)\).

Assume that the continuous Fourier transform \(F(\omega)\) is given. We are sampling it on the interval \((-R, R)\) transforming the continuous function to the equivalent discrete form. The number of the sample points on the interval \((0, R)\) is assumed to be \(N\); the total number of the points is \(NN = 2N\). under the above assumption, the sampling interval \(\Delta \omega\) is equal:

\[(5.7) \quad \Delta \omega = \frac{R}{N},\]

and, hence the sample points are defined as:

\[(5.8) \quad \omega_n = n \Delta \omega = n \frac{R}{N}, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \pm N.\]

By definition,

\[(5.9) \quad D_n = \bar{D}_{-n},\]

where the overbar denotes the complex conjugate of the number.
For the numerical calculations, the sample points of \( F(\omega) \) are located in the complex vector \( F \) of the length \( NN = 2N \) in a specific way. We substitute:

\[
F_1 = D_0, \; F_2 = D_1, \; \cdots , \; F_{N+1} = D_{\pm N}, \; F_{N+2} = D_{-N+1}, \; \cdots , \; F_{2N} = D_{-1},
\]

where the periodicity of the Fourier transform was assumed; the values of the transform at the ends of the interval of sampling are considered as equal – they are located at the element \( F_{N+1} \) of the discrete transform vector.

To restore the function \( f(t) \) we must calculate the discrete inverse Fourier transform defined as:

\[
f(t_m) = f_m = \frac{1}{T} \sum_{n=0}^{2N-1} D_n e^{i\omega_n t_m} = \frac{1}{T} \sum_{n=0}^{2N-1} D_n e^{2\pi n m / 2N},
\]

where the following definitions of \( t_m \) and \( \omega_n \) have been used:

\[
t_m = m \Delta T, \quad \omega_n = \frac{2\pi n}{2N \Delta T} = \frac{2\pi n}{T}.
\]

To connect the required quantity \( T \) describing time (the period of the function being transformed) with the dimension of the spectral domain of the transform we use the expressions for \( \omega_n \). Comparing \( \omega_n \) from (5.8) and (5.12) we obtain:

\[
\frac{n R}{N} = \frac{2\pi n}{2N \Delta T}.
\]

Then the time step in temporal discretization is:

\[
\Delta T = \frac{\pi}{R},
\]

and the period \( T \) is defined as:

\[
T = 2N \Delta T = \frac{2\pi N}{R}.
\]

For the numerical calculations of the discrete Fourier transform defined above we applied the procedure of the Fast Fourier Transform presented in the following reference [Press et al., 1986]. We have assumed in the calculations that the number of points \( N = 2^{16} = 65536 \).

As previously mentioned, in our numerical example we consider the convergence of the segmented bar to the homogenized case when the number of segments \( K \) tends to infinity. In contradiction to the harmonic waves where the reflection coefficient plays the role of a measure of the convergence (see [Kotulski, 1990, 1992]), in the case of pulses, defining an adequate measure is very difficult. Therefore, to show convergence, we present the shape of the reflected pulses for a given number of segments in the bar.
(under the constrain that the total share of the materials within the bar remains constant). In Figures 1-5 there are shown the amplitudes of the reflected pulse, respectively, for 1, 2, 4, 10, and 100 periodic couples of segments (figures marked with the letter a are for $Z_1 = 4.0$, $Z_2 = 2.0$, figures with b – for $Z_1 = 2.0$, $Z_2 = 4.0$). Figure 6 shows the reflected pulse when the bar is built of 500 periodic couples of segments (for such large number of segments the picture does not depend on the order of the materials). Finally, in Figure 7 the reflected pulse for the homogenized bar is presented (calculated according to the formula (3.16)).

Fig. 1. – The amplitude of the pulse reflected from the segmented bar composed of 1 couple of segments with the impedances. a) $Z_1 = 4.0$, $Z_2 = 2.0$; b) $Z_1 = 2.0$, $Z_2 = 4.0$.

Fig. 2. – As Figure 1 but for 2 couples of segments.
Fig. 3. – As Figure 1 but for 4 couples of segments.

Fig. 4. – As Figure 1 but for 10 couples of segments.

Fig. 5. – As Figure 1 but for 100 couples of segments.
The first conclusion we can draw from this sequence of plots is that the homogenization (the procedure regarded in theoretical considerations as an asymptotic phenomenon) really takes place. It is seen, that even for very small number of segments (e.g. 10) some concentration of pulses, similar to the case of the homogenized bar, takes place. In this case we can predict the location of the pulses in time but their amplitude still remains unknown. Increasing the density of stratification in the bar we reach a result close to the asymptotic limit. One can follow the speed of the convergence studying the presented set of pictures. It is also possible to observe how the localization of the wave pulse is generated: it is the result of summation of an infinitely growing number of pulses reflected and transmitted at the interfaces of the segments, (see Fig. 8).
Another conclusion observed from our calculations is that the reflected pulse for a small number of segments in the bar is strongly dependent on which material is the first in the periodic couple of segments. This dependence is stronger for the later reflected pulses. It practically disappears for a high number of segments in the bar.

Randomness of the segmentation of the bar makes the picture more complicated. In Figures 9 a, b there are two sample paths of the reflected wave pulse in the case of a random bar (built of, respectively, 10 and 100 couples of segments). In the calculations it is assumed that the lengths (travel times) of the segments are random variables ($H_i$ are i.i.d. random variables with the mean value equal to 1, uniformly distributed on (0,2)). We can see that in the random case the reflected pulses are much less regular; especially between the areas of concentration of pulses where there appears a number of small peaks (reflected pulses) which do not exist in deterministic case. They are more intensive for smaller $K$ – the number of segments in the bar; for increasing $K$ they should eventually disappear due to homogenization.

Fig. 9. – The sample-path of the amplitude of the reflected pulse in the case of randomly segmented bar, ($Z_1 = 2.0$, $Z_2 = 4.0$). a) The bar built of 10 segments; b) the bar built of 100 segments.

Figures 10 a, b show the averaged reflected amplitude of the wave pulse, obtained by the numerical simulation. It is seen that its shape is quite regular in comparison to both

Fig. 10. – The averaged amplitude of the reflected pulse in the case of randomly segmented bar, ($Z_1 = 2.0$, $Z_2 = 4.0$). a) The bar built of 10 segments; b) the bar built of 100 segments.

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the deterministic and the sample random amplitude. Moreover, for the averaged amplitude in between the maxima there are areas where it does not vanish. This permanent reflected pulse is the effect of averaging of the random peaks generated by individual sample paths of the reflected pulses. Such an effect decreases if the number of segments in the bar grows; it disappears for K tending to infinity.

Concluding, we observe that for the random segmentation of the bar the homogenization is much slower than in deterministic case. Moreover, the arrival time of the reflected pulse is difficult to detect because of the smooth shape of the amplitude. The effect of homogenization is weaker for the first pulse and stronger for later reflected pulses. This is the result of the greater number of reflections and transmissions affecting the pulses.

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