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On the moment stability of vibratory systems with random impulsive parametric excitation

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IN THE PAPER the mean and the mean square stability of the elastic column under an excitation of the form of random telegraph process is considered. Making use of the Shapiro-Loginov separation formula, a closed form of the moment equations is obtained both for systems governed by ordinary and partial differential equations. Application to the problem of stability is presented along with the comparison with the criterions known for a white noise and sinusoidal excitations. Analytical results are illustrated by numerical examples.

W pracy badana jest średnia i średniokwadratowa stabilność sprężystej kolumny poddanej działaniu wymuszenia w postaci losowego procesu telegraficznego. Przy użyciu formuły rozdzielania Szapiro-Loginowa otrzymano zamkniętą postać równań dla momentów, zarówno dla układów opisanych zwyczajnymi, jak i cząstkowymi równaniami różniczkowymi. Następnie przedstawiono zastosowanie tych równań do badania stabilności kolumny: uzyskane wyniki porównano z rezultatami znanymi dla wymuszenia białoszumowego i sinusoidalnego. Wyniki analityczne zilustrowano numerycznie dla realnych danych liczbowych.

В работе исследуется средняя и среднеквадратическая стабильность упругой колонны, подвергнутой действию возмущения в виде случайного телеграфного процесса. При использовании формулы разделения Шапиро-Логинова получен замкнутый вид уравнений для моментов, так для систем описанных обыкновенными уравнениями, как и дифференциальными уравнениями в частных производных. Затем представлено применение этих уравнений для исследования стабильности колонны; полученные результаты сравнены с известными результатами для белошумного и синусоидального возмущений. возмущениа Аналитические результаты иллюстрированы численно для реальных числовых данных.

1. Introduction

THE DYNAMIC stability of engineering systems under deterministic parametric excitation was extensively studied in the past. An investigation of the stability of various physical systems under stochastic excitation has also been performed using different definitions of the stochastic stability and different approaches (cf. [2, 4, 7, 10]). The existing results are, however, concerned mainly with excitations which vary randomly and continuously in time. To the best of our knowledge only a few papers have appeared in which the parametric excitation in the form of randomly arriving impulses have been considered (cf. [3, 5, 6, 8]).

It seems that the paper [8] by SAMUELS was the first one to discuss the dynamics of linear systems with coefficients forming a sequence of random impulses. The analysis presented in [8] adopts the idea of Foldy used in the multiple scattering of waves by randomly distributed point scatterers. The technique presented (under some restricting hypotheses) was applied to the problem of stability of circular shall under randomly arriving end thrusts. Paper [3] deals with the harmonic oscillator with the frequency being a two-valued Markov process, i.e. $\omega^2(t) = \omega_0^2[1+\varepsilon m(t)]$, where ε is a positive constant and $m(t)$ is a dichotomic Markov process with equiprobable values 1 and -1. The authors derived the expression for the mean square of stationary response and then, the mean square stability criterion. KLIATSKIN in his book [6] presents some averaging procedures for linear systems with randomly varying parameters, including parameters in the form of the random telegraph process and its generalization. However, the stability questions are not explicitly treated.

A systematic study of scalar differential equation of the form: $\dot{X}_t = h(X_t)+I_t g(X_t)$, where $X(t)$ is an unknown process, h and g are given, sufficiently smooth functions, and I_t is a random telegraph signal, has been provided in the book [5]. The authors have derived the Kolmogorov equation for the two-dimensional Markov process $[X_t, I_t]$, and then the equation for the probability density of the proces $X(t)$ alone, which takes the form of an integro-differential equation. The analogous equations have also been constructed — by different reasoning — in the book [6].

In the present paper we investigate the mean and mean-square stability of the response of linear systems with random telegraph process as a parametric excitation. Making use of the formula derived in [9], a closed form of moment equations is presented for systems governed by both the ordinary and partial differential equations. Afterwords, application to the problem of stability of an elastic column subjected to random telegraph type axial force is presented. The analytical results are illustrated numerically for real data.

2. General formulation

Let us consider a system of linear equations

$$\frac{dY}{dt} = \mathbf{A}Y+\mathbf{B}P(t,\gamma)Y, \tag{2.1}$$

$$Y(0) = Y_0,$$

where $Y(t) = [Y_1(t), \ldots, Y_n(t)]^T$ is an unown n-dimensional process, $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ deterministic matrix operators, Y_0 is a (deterministic) vector of initial conditions and $P(t,\gamma)$ is a random telegraph process ($\gamma\in\Gamma$, where Γ is the space of elementary events). It is assumed that $P(t,\gamma)$ is a scalar process.

According to its definition, process $P(t,\gamma)$ is characterized as

$$P(t,\gamma) = a(-1)^{N(t)}, \quad P(0,\gamma) = a, \quad P^2(t,\gamma) = a^2, \tag{2.2}$$

where a is a constant and $N(t)$ is a homogeneous Poisson process with intensity ν. It can be shown that the following reccurent relations hold

$$m_n(t_1, \ldots, t_n) = \langle P(t_1,\gamma) \ldots P(t_n,\gamma)\rangle$$
$$= \langle P(t_1,\gamma)P(t_2,\gamma)\rangle m_{n-2}(t_3, \ldots, t_n), \quad (t_1 \geqslant t_2 \geqslant \ldots \geqslant t_n > 0), \tag{2.3}$$

where $\langle\cdot\rangle$ denotes the probabilistic averaging (the average with respect to probability measure). Two moments of $P(t,\gamma)$ of the lowest order are

$$\langle P(t,\gamma)\rangle = ae^{-2\nu t}, \quad \langle P(t_1,\gamma)P(t_2,\gamma)\rangle = a^2e^{-2\nu|t_1-t_2|}. \tag{2.4}$$

Let $R_t[P(\tau)]$ be a functional depending on the values of $P(\tau)$ for $\tau < t$. It has been shown in [9] (cf. also [6]) that

$$\frac{d}{dt}\langle P(t)R_t[P]\rangle = \left\langle P(t)\frac{d}{dt}R_t[P]\right\rangle - 2\nu\langle P(t)R_t[P]\rangle. \tag{2.5}$$

The above formula for splitting correlations allows to obtain a closed system of equations for $\langle Y(t)\rangle$. Indeed, taking the average of both sides of equation (2.1) we have

$$\left\langle\frac{dY}{dt}\right\rangle = \mathbf{A}\langle Y\rangle + \mathbf{B}\langle P(t,\gamma)Y\rangle. \tag{2.6}$$

To find the equation for $\langle P(t,\gamma)Y\rangle$ use is made of formula (2.5); it should be noticed that $Y(t)$, as a solution of (2.1), depends on the values of $P(\tau,\gamma)$ only for $\tau \leqslant t$. Differentiation with respect to t results in

$$\frac{d}{dt}\langle P(t,\gamma)Y\rangle = \langle P(t,\gamma)\dot{Y}\rangle - 2\nu\langle P(t,\gamma)Y\rangle, \qquad \dot{Y} = \frac{dY}{dt}.$$

Substituting $\dot{Y}$ for the expression taken from the governing equation (2.1) gives a closed equation for $\langle Z(t)\rangle = \langle P(t,\gamma)Y(t)\rangle$. Thus we have the following system of equations for the mean solution

$$\begin{aligned}\frac{d}{dt}\langle Y\rangle &= \mathbf{A}\langle Y\rangle + \mathbf{B}\langle Z\rangle,\\ \frac{d}{dt}\langle Z\rangle &= (\mathbf{A}-2\nu\mathbf{I})\langle Z\rangle + a^2\mathbf{B}\langle Y\rangle\end{aligned} \tag{2.7}$$

(I is $n\times n$ unit matrix).

The initial conditions associated with system (2.7) are

$$Y(0) = Y_0, \qquad Z(0) = \langle P(0,\gamma)Y(0)\rangle = aAY_0. \tag{2.8}$$

Formula (2.5) makes it also possible to obtain a closed system of equations for the second order moments $\langle YY^T\rangle$. This system has the form ($\zeta = P(t,\gamma)YY^T$):

$$\begin{aligned}\frac{d}{dt}\langle YY^T\rangle &= \mathbf{A}\langle YY^T\rangle + \langle YY\rangle\mathbf{A} + \mathbf{B}\langle\zeta\rangle + \langle\zeta\rangle\mathbf{B}^T,\\ \frac{d}{dt}\langle\zeta\rangle &= \mathbf{A}\langle\zeta\rangle + \langle\zeta\rangle\mathbf{A}^T - 2\nu\langle\zeta\rangle + a^2\mathbf{B}\langle YY^T\rangle + a^2\langle YY^T\rangle\mathbf{B}^T,\end{aligned} \tag{2.9}$$

where superscript T denotes the transposition of the matrix concerned.

The deterministic systems (2.7) and (2.9) form the base for deriving the conditions of mean and mean-square stability. It should be noticed that equations (2.7), (2.9) hold for the general case when matrix operators $\mathbf{A}$ and $\mathbf{B}$ depend on time. In the specific problem considered below these matrices are assumed to be constant in time.

The application of formula (2.5) can be extended to the systems of partial differential equations. Let the system of equations have the form

$$\frac{\partial u_i(x,t)}{\partial t} = \sum_{j=1}^{k}\mathbf{A}_{ij}(x)u_j + P(t,\gamma)\sum_{j=1}^{k}\mathbf{B}_{ij}(x)u_j, \tag{2.10}$$

where $x \in \mathbf{R}^n$, $t \in [0,\infty)$, $i = 1, 2, \ldots, k$, and $\mathbf{A}_{ij}(x)$, $\mathbf{B}_{ij}(x)$ are differential operators with respect to x; $i, j = 1, 2, \ldots, k$.

The closed system of equations for the mean solution is as follows ($Z_j = P(t,\gamma)u_j(x,t)$):

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle u_i\rangle &= \sum_{j=1}^{k}\mathbf{A}_{ij}(x)\langle u_j\rangle+\sum_{j=1}^{k}\mathbf{B}_{ij}(x)\langle Z_j\rangle,\\
\frac{\partial}{\partial t}\langle Z_i\rangle &= \sum_{j=1}^{k}[\mathbf{A}_{ij}(x)-2\nu\delta_{ij}]\langle Z_j\rangle+\sum_{j=1}^{k}\mathbf{B}_{ij}(x)\langle Z_j\rangle\\
&\quad+a^2\sum_{j=1}^{k}\mathbf{B}_{ij}(x)\langle u_j\rangle,\qquad i=1,2,\ldots,k.
\end{aligned}
\tag{2.11}
$$

Let us denote

$$
\begin{aligned}
Q_{ij}(x,y,t) &= \langle u_i(x,t)u_j(y,t)\rangle,\\
R_{ij}(x,y,t) &= \langle P(t,\gamma)u_i(x,t)u_j(y,t)\rangle.
\end{aligned}
$$

The equations for the second order moments are

$$
\begin{aligned}
\frac{\partial}{\partial t}Q_{ij}(x,y,t) &= \sum_{l=1}^{k}\mathbf{A}_{il}(x)Q_{lj}(x,y,t)+\sum_{l=1}^{k}\mathbf{A}_{jl}(y)Q_{il}(x,y,t)\\
&\quad-2\nu Q_{ij}(x,y,t)+\sum_{l=1}^{k}\mathbf{B}_{il}(x)R_{lj}(x,y,t)+\sum_{l=1}^{k}\mathbf{B}_{jl}(y)R_{il}(x,y,t),\\
\frac{\partial}{\partial t}R_{ij}(x,y,t) &= \sum_{l=1}^{k}\mathbf{A}_{il}(x)R_{lj}(x,y,t)+\sum_{l=1}^{k}\mathbf{A}_{jl}(y)R_{il}(x,y,t)\\
&\quad-2\nu R_{ij}(x,y,t)+a^2\sum_{l=1}^{k}\mathbf{B}_{il}(x)Q_{lj}(x,y,t)+a^2\sum_{l=1}^{k}\mathbf{B}_{jl}(y)Q_{il}(x,y,t),\\
&\qquad i,j=1,2,\ldots,k.
\end{aligned}
\tag{2.12}
$$

The above systems of equations can be used to estimate the domains of mean-square stability of continuous structural systems governed by partial differential equations with parameters fluctuating according to random telegraph signal.

3. Column under random telegraph axial excitation

3.1. Stability of the fundamental mode

Let us consider an uniform pin-ended column of length L subjected to a stream of pulses which act axially and randomly in time in accordance to the random telegraph process $p(t,\gamma)$. The small transverse motion of the column is governed by the equation (cf. [1])

$$
EI\frac{\partial^4 u}{\partial x^4}+p(t,\gamma)\frac{\partial^2 u}{\partial x^2}+\varrho\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial u}{\partial t}=0
\tag{3.1}
$$

with the boundary conditions:

$$
u=\frac{\partial^2 u}{\partial x^2}=0\quad\text{at}\quad x=0,L.
$$

In Eq. (3.1) $u(x, t)$ is transverse deflection of a column, EI — its flexural rigidity, ϱ — mass per unit length, β — coefficient of viscous damping per unit length.

Using the model approach and looking for the transverse motion in the first (fundamental) mode, that is

$$u(x, t) = y(t)\sin\frac{\pi x}{L},$$

the deflection of the mid-section $y(t)$ satisfies the equation

$$\ddot{y}+\frac{\beta}{\varrho}\dot{y}+\frac{\pi^2}{\varrho L^2}(P_E-p(t, \gamma))y = 0, \tag{3.2}$$

where $P_E = \frac{n^2 EI}{L^2}$ is the Euler load. Let the axial force $p(t, \gamma)$ have the form

$$p(t, \gamma) = P_0\big(1+a_0(-1)\big)^{N(t, \gamma)}, \qquad P_0 < P_E. \tag{3.3}$$

Equation (3.2) takes now the form

$$\ddot{y}+2h\dot{y}+\omega_0^2\big(1-P(t, \gamma)\big)y = 0, \tag{3.4}$$

where

$$h = \frac{\beta}{2\varrho}, \qquad \omega_0^2 = \frac{\pi^2}{\varrho L^2}(P_E-P_0), \qquad P(t, \gamma) = a(-1)^{N(t, \gamma)}, \tag{3.5}$$

and

$$a = \frac{P_0 a_0}{P_E-P_0}.$$

A system of the first order equations corresponding to (3.4) is (let $y(t) = y_1(t)$):

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= -2hy_2(t)-\omega_0^2 y_1(t)-\omega_0^2 P(t, \gamma)y_1(t). \end{aligned} \tag{3.6}$$

According to the notations used in Sect. 2, $Y = [y_1, y_2]^T$ and $Z = [z_1, z_2]^T = [P(t, \gamma)y_1, P(t, \gamma)y_2]^T$. System (2.7) for the mean solution takes the form

$$\begin{aligned} \frac{d}{dt}\langle y_1\rangle &= \langle y_2\rangle, \\ \frac{d}{dt}\langle y_2\rangle &= -2h\langle y_2\rangle-\omega_0^2\langle y_1\rangle-\omega_0^2\langle z_1\rangle, \\ \frac{d}{dt}\langle z_1\rangle &= -2\nu\langle z_1\rangle+\langle z_2\rangle, \\ \frac{d}{dt}\langle z_1\rangle &= -2h\langle z_2\rangle-2\nu\langle z_2\rangle-\omega_0^2\langle z_1\rangle-a^2\omega_0^2\langle y_1\rangle. \end{aligned} \tag{3.7}$$

System (3.7) is of the form $\frac{d}{dt}\eta = \mathscr{A}\eta$, where

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_0^2 & -2h & -\omega_0^2 & 0 \\ 0 & 0 & -2\nu & 1 \\ -\alpha^2\omega_0^2 & 0 & -\omega_0^2 & -2h-2\nu \end{bmatrix}. \tag{3.8}$$

The use of the Ruth–Hurwitz criterion yields the conditions for the mean stability. The characteristic equation is

$$\det(\mathscr{A}-\lambda I) = \lambda^4+\alpha_3\lambda^2+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0 = 0, \tag{3.9}$$

where

$$\begin{aligned}\alpha_0 &= \omega_0^2[\omega_0^2(1-a^2)+4\nu(h+\nu)],\\ \alpha_1 &= 4(h+\nu)(2h\nu+\omega_0^2),\\ \alpha_2 &= 2_0^2\omega+4(h^2+3h\nu+\nu^2),\\ \alpha_3 &= 4(h+\nu).\end{aligned} \tag{3.10}$$

The roots of equation (3.9) have negative real parts if and only if, simultaneously,

$$\alpha_0>0, \quad W_1 = \alpha_1>0, \quad W_2 = \alpha_1\alpha_2-\alpha_0\alpha_3>0, \quad W_3 = W_2\alpha_3-\alpha_1^2>0.$$

It can be easily verified that the first condition gives

$$a^2 < \frac{4\nu^2+4h\nu+\omega_0^2}{\omega_0^2}, \tag{3.11}$$

whereas the remaining conditions are fulfilled automatically. Therefore, the conclusion is: the column is stable in the mean if Eq. (3.11) holds. This gives the following condition for the intensity of the Poisson stream of pulses

$$\frac{4\varrho L^2}{\pi^2}\nu^2+\frac{2\beta L^2}{\pi^2}\nu-\frac{P_0^2a_0^2-(P_E-P_0)^2}{P_E-P_0} > 0. \tag{3.12}$$

A graphical illustration of the above condition will be given in the next section.

The equations governing the second moments $\langle y_1^2\rangle$, $\langle y_1y_2\rangle$, $\langle y_2^2\rangle$, $\langle\zeta_1\rangle=\langle Py_1^2\rangle$, $\langle\zeta_2\rangle=\langle Py_1y_2\rangle$, $\langle\zeta_3\rangle=\langle Py_2^2\rangle$ are as follows

$$\begin{aligned}
\frac{d}{dt}\langle y_1^2\rangle &= 2\langle y_1y_2\rangle,\\
\frac{d}{dt}\langle y_1y_2\rangle &= \langle y_2^2\rangle-2h\langle y_1y_2\rangle-\omega_0^2\langle y_1^2\rangle-\omega_0^2\langle\zeta_1\rangle,\\
\frac{d}{dt}\langle y_2^2\rangle &= -4h\langle y_2^2\rangle-2\omega_0^2\langle y_1y_2\rangle-2\omega_0^2\langle\zeta_2\rangle,\\
\frac{d}{dt}\langle\zeta_1\rangle &= -2\nu\langle\zeta_1\rangle+2\langle\zeta_2\rangle,\\
\frac{d}{dt}\langle\zeta_2\rangle &= -2(h+\nu)\langle\zeta_2\rangle+\langle\zeta_3\rangle-\omega_0^2\langle\zeta_1\rangle-\omega_0^2a^2\langle y_1^2\rangle,\\
\frac{d}{dt}\langle\zeta_3\rangle &= -2(2h+\nu)\langle\zeta_3\rangle-2\omega_0^2\langle\zeta_2\rangle-2\omega_0^2a^2\langle y_1y_2\rangle.
\end{aligned} \tag{3.13}$$

The characteristic equation associated with system (3.13) is

$$\lambda^6+\delta_5\lambda^5+\delta_4\lambda^4+\delta_3\lambda^3+\delta_2\lambda^2+\delta_1\lambda+\delta_0=0,$$

where

$\delta_5 = 6(2h+\nu)$,

$\delta_4 = 4(2\omega_0^2+15\nu h+13h^2+3\nu^2)$,

$$\delta_3 = 32\omega_0^2(2h+\nu)+8(12h^3+12\nu^2h+26h^2\nu+\nu^3),$$
$$\delta_2 = 16[\omega_0^4(1-a^2)+\omega_0^2(10h^2+3\nu^2+12h\nu)+h(4h^3+15h\nu^2+18h^2\nu+3\nu^3)],$$
$$\delta_1 = 32[\omega_0^4(2h+\nu)(1-a^2)+\omega_0^2(4h^3+10h^2\nu+6h\nu^2+\nu^3)+2h^2\nu(2h^2+\nu^2+3h\nu)],$$
$$\delta_0 = 16\omega_0^2[4h(h+\nu)\omega_0^2+4\nu h(h+\nu)(2h+\nu)-a^2(2h+\nu)^2\omega_0^2].$$

Application of the Ruth–Hurwitz criterion gives the following conditions for the mean — square stability:

$$\begin{aligned} &\delta_0 > 0, \quad \delta_1 > 0, \quad M_2 = \delta_1\delta_2-\delta_3\delta_0 > 0,\\ &M_3 = \delta_3 M_2-\delta_1(\delta_1\delta_4-\delta_0\delta_5) > 0,\\ &M_4 = \delta_4 M_3-\delta_5[\delta_2 M_2-\delta_0(\delta_1\delta_4-\delta_0\delta_5)] > 0,\\ &M_5 = \delta_5 M_4-\delta_3 M_3+\delta_1(\delta_5 M_2-\delta_1^2) > 0.\end{aligned} \tag{3.14}$$

The graphical illustration of the above conditions obtained from the numerical calculations will be given in the next section.

3.2. Complete transverse response; the moment equations

A more satisfactory approach to the analysis of stability of continuous structural systems deals with the solution of the appropriate partial differential equation (without representing it in the form of a modal equation). In the case of a column subjected to random telegraph axial force the moment stability can, in principle, be derived from general equations (2.11) and (2.12). Indeed, representing the governing equation (3.1) in the form of the system (let $u(x,t) = u_1(x,t)$)

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = u_2(x,t),\\ &\frac{\partial u_2}{\partial t} = -\frac{EI}{\varrho}\frac{\partial^4 u_1}{\partial x^4}-\frac{P_0}{\varrho}\frac{\partial^2 u_1}{\partial x^2}-\frac{P_0}{\varrho}P_1(t,\gamma)\frac{\partial^2 u_1}{\partial x^2}-\frac{\beta}{\varrho}u_2,\end{aligned} \tag{3.15}$$

where

$$P_1(t,\gamma) = \alpha_0(-1)^{N(t,\gamma)},$$

equations (2.11) and (2.12) for the mean and the mean-square, respectively, may be applied. In the case of a column, operators $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ are

$$\begin{aligned} &\mathbf{A}_{11}(x) = 0, \quad \mathbf{A}_{12}(x) = 1, \quad \mathbf{A}_{21}(x) = -\frac{EI}{\varrho}\frac{\partial^4}{\partial x^4}-\frac{P_0}{\varrho}\frac{\partial^2}{\partial x^2},\\ &\mathbf{A}_{22}(x) = -\frac{\beta}{\varrho},\\ &\mathbf{B}_{11}(x) = \mathbf{B}_{12}(x) = 0, \quad \mathbf{B}_{21}(x) = -\frac{P_0}{\varrho}\frac{\partial^2}{\partial x^2}, \quad \mathbf{B}_{22}(x) = 0.\end{aligned} \tag{3.16}$$

Therefore, the equations for the mean are the following:

$$\begin{aligned} &\frac{\partial}{\partial t}\langle u_1\rangle = \langle u_2\rangle,\\ &\frac{\partial}{\partial t}\langle u_2\rangle = -\frac{EI}{\varrho}\frac{\partial^4\langle u_1\rangle}{\partial x^4}-\frac{P_0}{\varrho}\frac{\partial^2\langle u_1\rangle}{\partial x^2}-\frac{P_0}{\varrho}\frac{\partial^2\langle Z_1\rangle}{\partial x^2}-\frac{\beta}{\varrho}\langle u_2\rangle,\end{aligned} \tag{3.17}$$

$$\frac{\partial}{\partial t}\langle Z_1\rangle = -2\nu\langle Z_1\rangle+\langle Z_2\rangle, \tag{3.17}$$
[cont.]
$$\frac{\partial}{\partial t}\langle Z_2\rangle = -\frac{EI}{\varrho}\frac{\partial^4\langle Z_1\rangle}{\partial x^4}-\frac{P_0}{\varrho}\frac{\partial^2\langle Z_1\rangle}{\partial x^2}-\frac{a^2P_0}{\varrho}\frac{\partial^2\langle u_1\rangle}{\partial x^2}-\left(\frac{\beta}{\varrho}+2\nu\right)\langle Z_2\rangle.$$

Analogously, one can write the system of eight equations for the second-order moments. The systems of partial differential equations for the first and second order moments (in the case considered) are rather complicated and the stability regions — if necessary — can be estimated numerically.

3.3. Relation to white noise excitation

The correlation time of a random telegraph process, what is seen from (2.4), is $1/2\nu$ and tends to zero as the intensity ν in the Poisson stream of pulses tends to infinity. This is a heuristic explanation of the fact that process $P(t, \gamma)$ tends to a white noise (as $\nu\to\infty$ but a^2/ν remains finite). Therefore, one could expect that for very large ν the range of stochastic stability for random telegraph excitation should be close to that corresponding to white noise excitation with the "equivalent" intensity $I = a^2/\nu$.

Let us assume that the process $P(t, \gamma)$ in (3.4) is replaced by a white noise with intensity $I = a^2/\nu$. As it was shown in paper [1], the mean stability takes place for all possible values of parameters involved. The condition of the mean square stability is

$$\frac{a^2}{\nu} < \frac{2\beta L^2}{\pi^2P_0^2}(P_E-P_0), \tag{3.18}$$

and it is much simpler than the analogous condition for the excitation of the telegraph type. After substitution of the parameter a into (3.18) we obtain the stability condition in the final form:

$$\frac{1}{\nu} < \frac{2\beta L^2}{\pi^2P_0^4a_0^2}(P_E-P_0)^3.$$

A graphical comparison of these two conditions will be given in the next section.

3.4. Relation to sinusoidal excitation

We can imagine the telegraph symmetric stochastic process as a process which in randomly occuring instants of time jumps between its two states. Such a situation is quite similar to a sinusoidal excitation (with some, possibly random, frequency ω).

The conditions of stability of the column under axial sinusoidal excitation has been investigated in the literature. If the force in (3.1) is deterministic with frequency ω, that is, if it is of the form

$$p(t) = P_0(1+a_1\sin\omega t), \tag{3.19}$$

then the conditions of stability of the fundamental mode has the form (see [10]):

$$\omega < -\frac{2\beta\varkappa(1+\alpha)^2}{\varrho\alpha\left[\frac{\beta^2}{\varrho^2}+\varkappa^2(1+\alpha)^2\right]},$$
(3.20)
$$\omega\left(\frac{\alpha\beta}{2\varrho\varkappa(1+\alpha)^2}+\frac{\varkappa\alpha\varrho}{2\beta}+\frac{\alpha\varrho}{2\beta\varkappa(1+\alpha)^2}+\frac{\alpha^2\varrho^2\omega}{4\beta^2(1+\alpha)^2}\right) < -1, \quad |\alpha|<1,$$

(along with the obvious conditions: $\beta > 0$, $P_E > P_0$), where

$$\varkappa = \frac{1}{\varrho}\frac{\pi^2}{L^2}[P_E - P_0], \qquad \alpha = \frac{a_1 P_0}{P_E - P_0}.$$

To compare the regions of stability for the sinusoidal and telegraphic excitations we should choose some criterions of equivalence of the processes. A possible criterion could be that both processes should spend (in the mean) the same time above the axis during one cycle. Since for the Poisson process $N(t, \gamma)$ in (3.3) the average time between jumps is $1/\nu$ and half of the period of the sinus function is π/ω, the frequency of the "equivalent" sinusoidal excitation should be:

$$\omega = \pi\nu. \tag{3.21}$$

A suplementary criterion needed for comparison of the effects of random telegraph and sinusoidal excitations is the reqiurement that the impulses of the exciting forces during one period are equal,

$$\int_0^{1/\nu} a_1 \sin(\pi\nu\tau)\, d\tau = a_0/\nu,$$

what results in

$$a_1 = \frac{\pi}{2} a_0. \tag{3.22}$$

Since the first of the conditions (3.20) is valid only for negative frequencies, in order to obtain the corresponding relationships for the equivalent intensity (which is always positive), we should put in (3.20) $-\pi\nu$ instead of ω. The resulting inequalities are

$$\begin{gathered} \nu > \frac{2\beta\varkappa(1-\alpha)^2}{\varrho\pi\alpha\left[\frac{\beta^2}{\varrho^2} + \varkappa^2(1-\alpha)^2\right]}, \\ \nu\left(\frac{\alpha\beta}{2\varrho\varkappa(1-\alpha)^2} + \frac{\varkappa\alpha\varrho}{2\beta} + \frac{\alpha\varrho}{2\beta\varkappa(1+\alpha)^2} - \frac{\alpha^2\varrho^2\nu\pi}{4\beta^2(1+\alpha)^2}\right) > \frac{1}{\pi}, \\ |\alpha| < 1, \end{gathered} \tag{3.23}$$

where $\varkappa$ is the same as in (3.20), and

$$\alpha = \frac{a_0 P_0 \pi}{2(P_E - P_0)}.$$

Region of stability described by conditions (3.23) will be compared with the stability condition for telegraphic excitation graphically in the next section.

4. Numerical calculations; conclusions

The formulae describing the conditions of stability are rather complicated. To compare the regions of stability for the excitation processes considered above and to present them graphically, let us present the numerical examples.

Consider a column of circular cross-section made of steel. The material constants are

$$\text{Youngs modulus} \qquad E = 2\times 10^{10}\ \text{kG/m}^2,$$
$$\text{material density} \qquad d = 7880\ \text{kg/m}^3.$$

Dimensions of the sample structure are: length (L) is 10 meters and radius of the cross-section is 0.1 m. Therefore the constant necessary for numerical calculations are:

$$\text{flexural rigidity} \qquad EI = \pi/2\times 10^6 \simeq 1.57\times 10^6\ (\text{kGm}^2),$$
$$\text{mass per unit length} \qquad \varrho = 78.8\pi = 247.6\ (\text{kg/m}).$$

The critical Eulerian load for such a column is

$$P_E = 1.5495\times 10^5\ (\text{kG}).$$

The remaining parameters describe external excitation (P_0, a_0, ν) and properties of the environment (β). Plotting the areas of stability in the $\nu-\beta$ coordinate system we fix the constants P_0 and a_0 in such a way (different in each case of stability criterion) that the obtained sets are nontrivial.

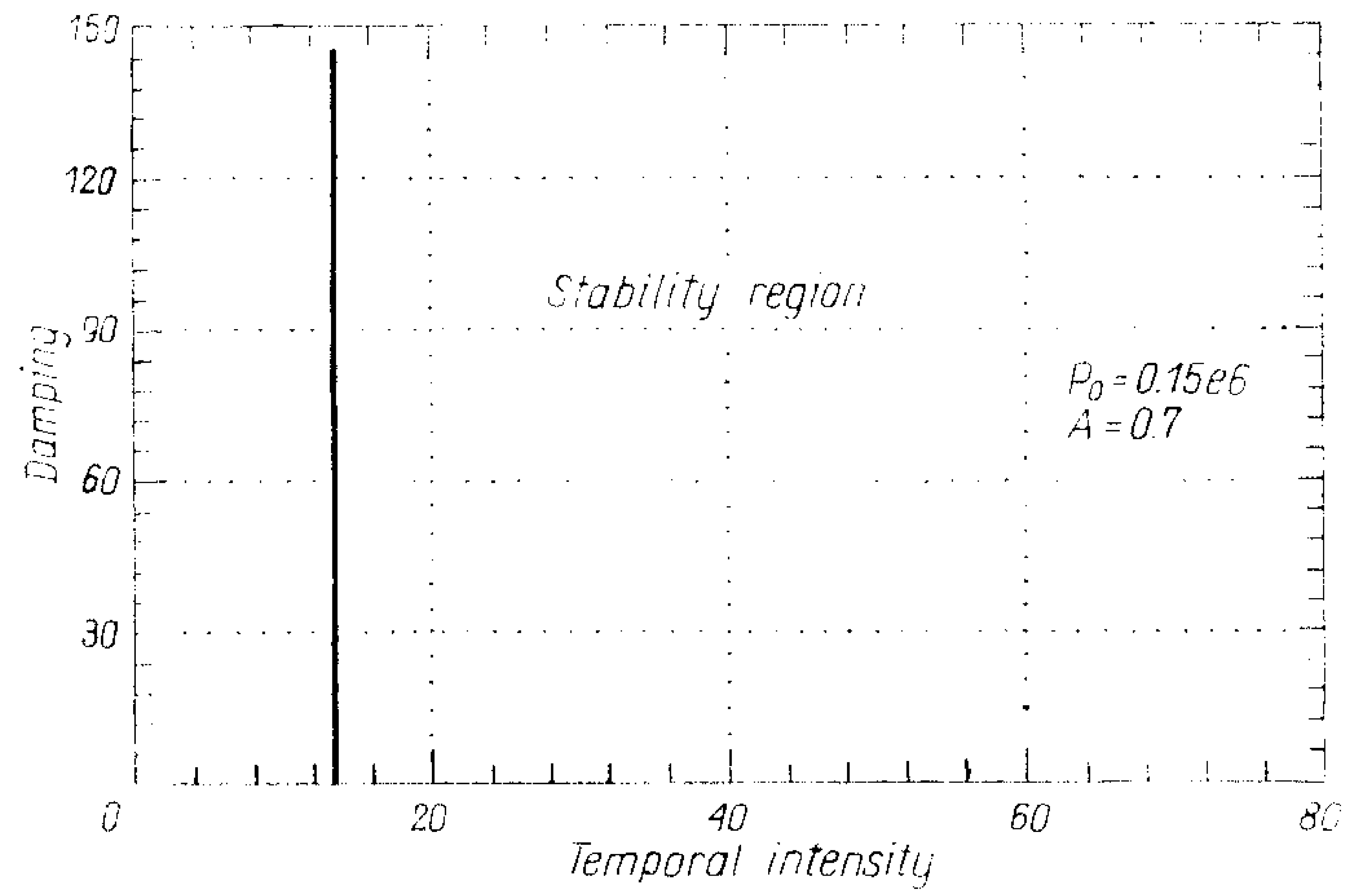

FIG. 1. Stability in the mean (the first mode approximation). Random telegraphic excitation.

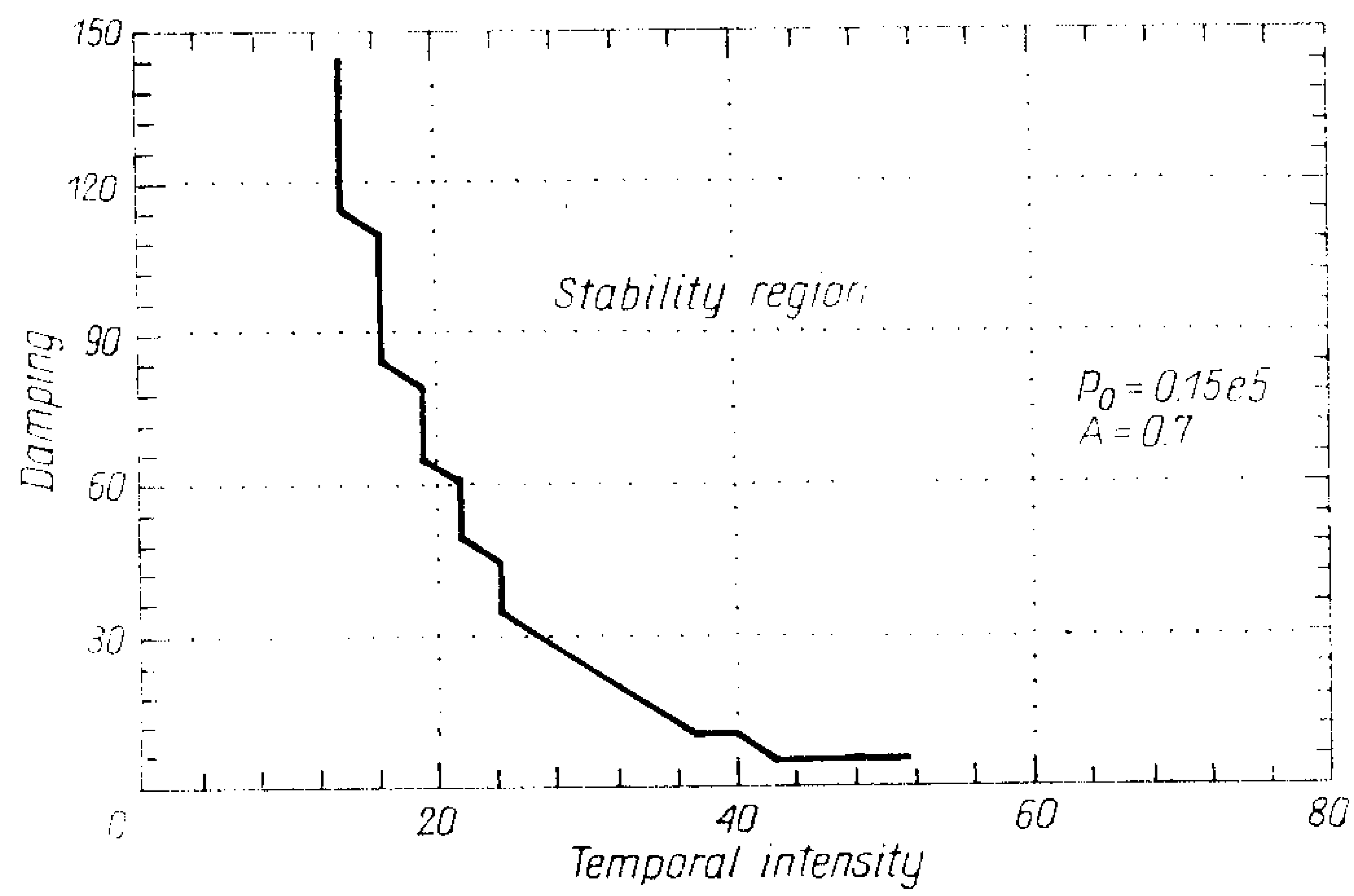

FIG. 2 The mean square stability (the first mode approximation). Random telegraphic excitation.

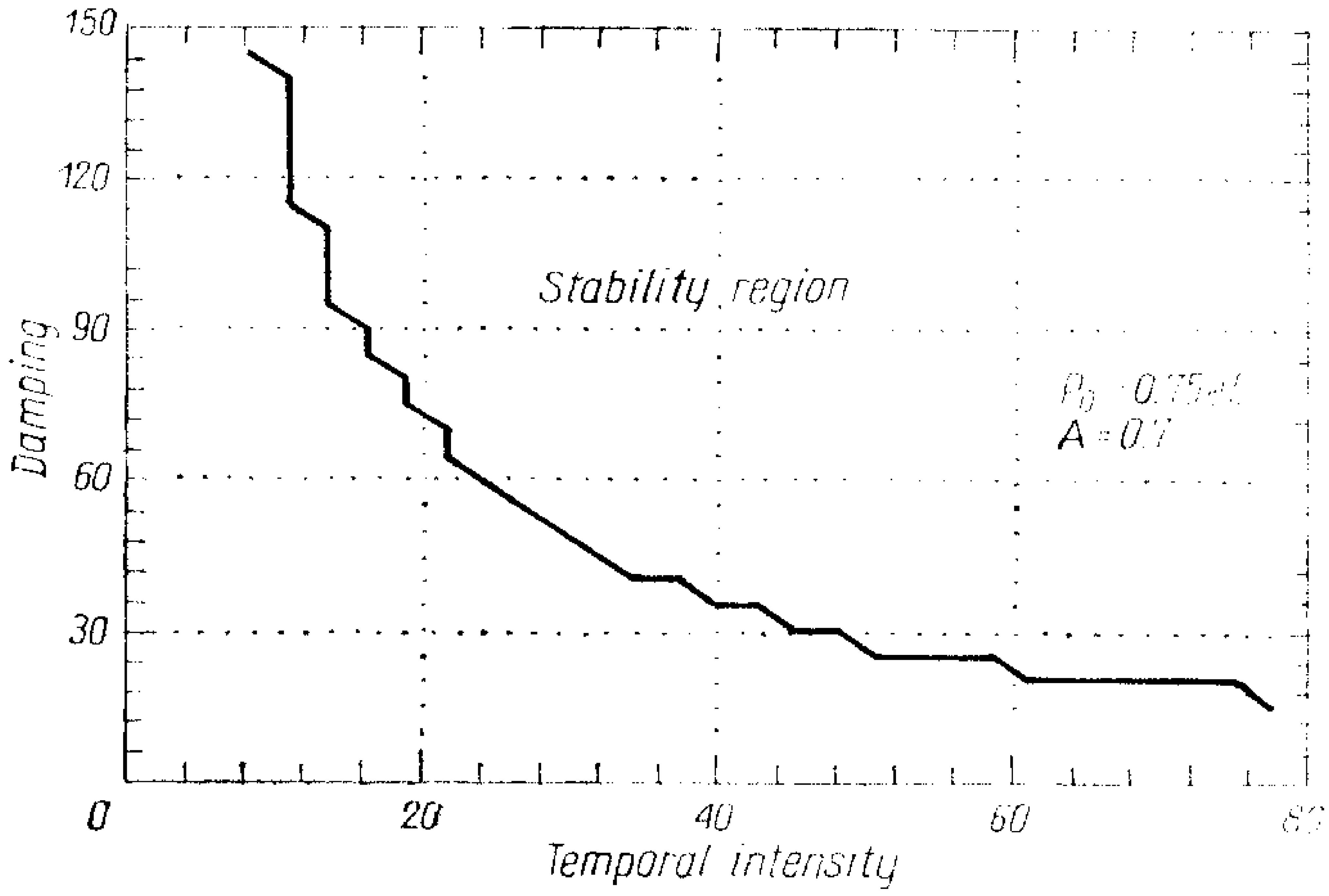

FIG. 3. The mean square stability (the first mode approximation) White-noise excitation.

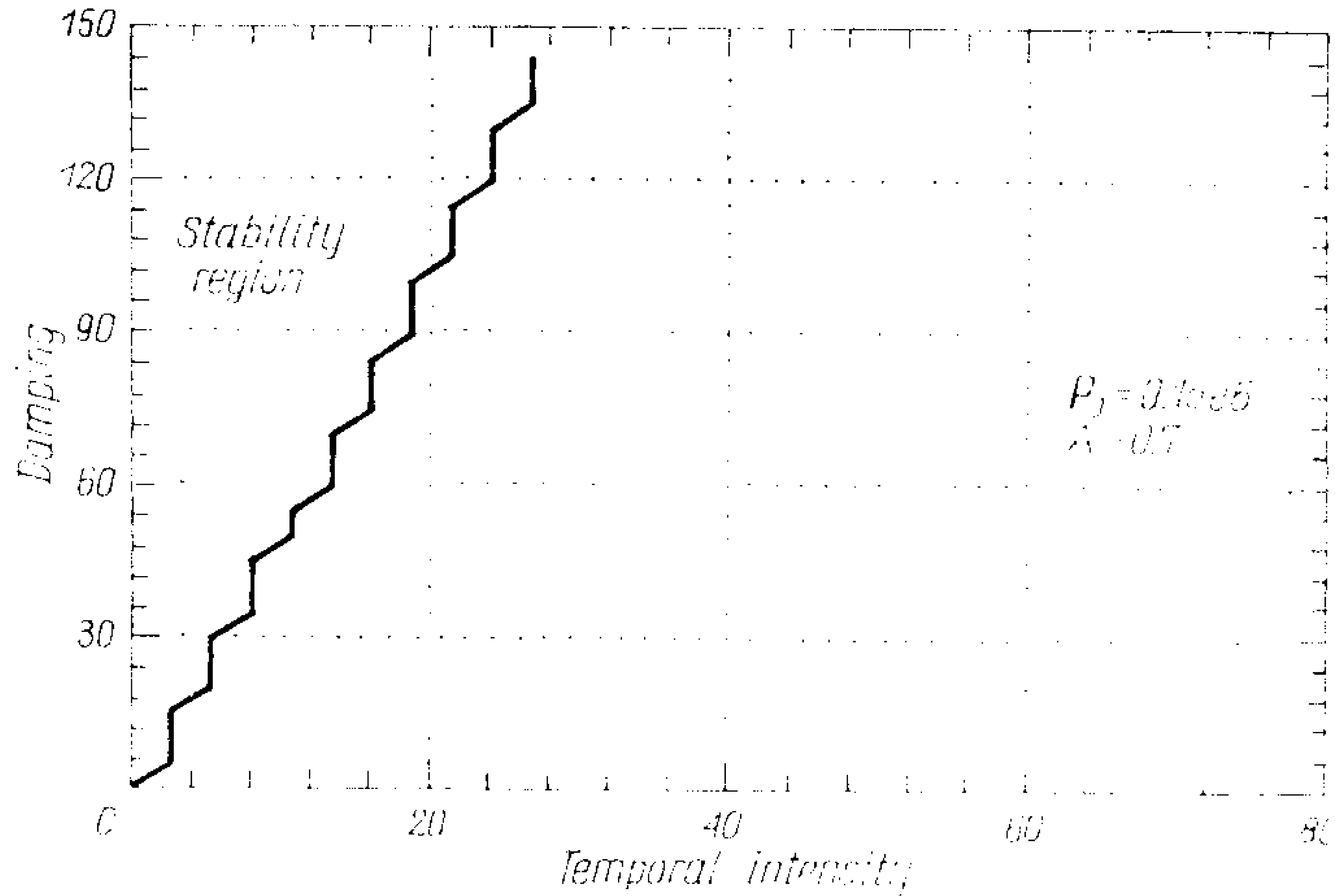

FIG. 4. Stability of deterministic system (the first mode approximation). Sinusoidal excitation.

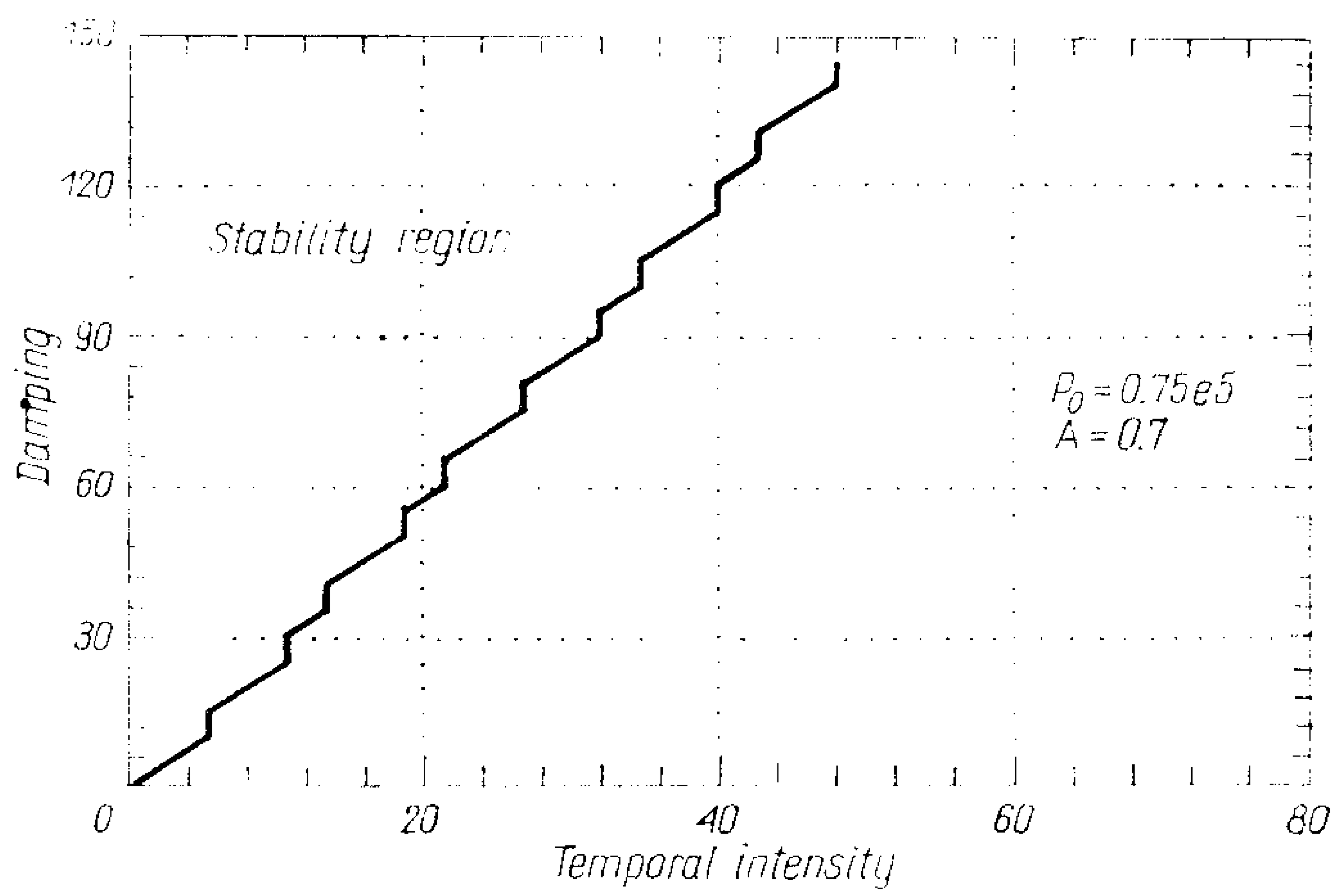

FIG. 5. Stability of deterministic system (the first mode approximation). Sinusoidal excitation.

In Fig. 1 the values of ν and β are shown for which the column is stable in the mean. The parameters are chosen $P_0 = 15\times 10^6$, $A = 0.7$. For such values P_0 and A the column is always unstable in the mean square. For the above quantities also the "equivalent" white noise excitation makes the column mean square unstable.

If we assume $P_0 = 5\times 110$, $A = 0.7$, then the column is always stable in the mean, stable for white noise excitation except for $\beta = 0$ and $\nu = 0$; the regions of the mean square stability under the telegraph excitation are shown in Fig. 2.

Figure 3 shows that the region of mean square stability for the white noise excitation is quite similar to that in Fig. 2 for the telegraphic one; however, the range of the force in Fig. 3 is five times greater and in such a case the telegraphic excitation destabilizes the system.

The stability region for the corresponding sinusoidal excitation differs significantly from the one in telegraphic or white noise case (see Fig. 4 and 5). However, we should keep in mind that conditions (3.11), (3.14) and (3.18) are necessery and sufficient but condition (3.23) is only sufficient.

As a conclusion of the above considerations we can say that in approximating one of the studied types of excitation processes by another we must be very careful. The verification if both of them are stable for the given parameters P_0, A, β and ν is always necessary.

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