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Wave Propagation in Randomly Stratified Media and Anderson Localization

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Abstract - In the paper the phenomenon of Anderson localization is described in context of wave propagation in randomly stratified media. It is shown, how the localization is generated both in stationary and non stationary cases.

1. THE ANDERSON MODEL OF LOCALIZATION

In many periodic physical systems it is possible transport of some disturbances, e.g. mechanical waves and vibrations in structures, electromagnetic waves in stratified plasma, e.t.c. Since in such systems it is assumed no damping, the disturbance can propagate unlimited. However, it was observed that when the perfect periodicity of the system is randomly perturbed, the possibility of the propagation disappears and the effects of initial disturbance remain within some bounded region of space. In contradiction to dispersion phenomenon, the initially propagating object does not change its character during localization, for example the energy of acoustic wave is not transformed into heat.

The phenomenon of localization has been studied for the first time by P.W.Anderson [1]. He considered the model of quantum particles (electrons) transport in a crystal lattice. In the case when the lattice is perfectly periodic the particle is diffunding unlimited, independently of its energetic state (all the states of the particle are extended; they are Bloch states). This means, that the probability that the particle reaches any site in the lattice is independent of the site and the eigenstates are distributed over all the lattice. In the case of the randomly disordered lattices, the localized states of the particles can appear. This means that for some states, after infinitely long time, the particles can be found in a finite surrounding of the starting point with non-zero probability. In other words, the mean square distance travelled by the particle up to time *t* is bounded uniformly in time. In such a case the energy eigenstates (wave functions) are exponentially bounded behind limited surrounding of some site in the lattice.

The fact, if the particle state is extended or localized depends on the order of randomness of the lattice. For weakly disordered lattices there are possible both extended and localized states. In such lattices the transport of particles is possible, but is less intensive than in perfectly periodic lattices; moreover, inside the random lattice there are more particles than in analogous volume of the perfectly periodic one. For strongly disordered lattices all the particle states are localized and no (global) transport is possible.

The localization phenomenon is very well physically motivated. However, it is also in the area of interest of mathematicians [2]. There were many mathematical theories and tools applied to describe and analyse the localization models. Some of them are the random potential theory [3], random Shrödinger [4], [5] and self-adjoint [6] operators, Lyapunov exponents [7], limit theorems for non-commuting products [8], [9] and others. More information about the mathematical methods concerned with the Anderson localization can be found the references cited in [2-7].

2. EXAMPLES OF THE LOCALIZATION IN OTHER PHYSICAL SYSTEMS

The paper of Anderson initiated a new area of investigation in mathematical physics and applied mathematics - studies on the localization phenomenon. In the literature there is a large number of papers where the authors identified the effects analogous to Anderson localization both in artificially imagined mechanical systems and in the models of really existing physical phenomena. In this section we give several examples of various randomly disturbed periodic systems where the localization has been observed. Especially one dimensional models are easy to deduce the localization. In such a case even very small deviation from the ideal periodicity makes that the propagation is localized. For this reason the most of the papers deal with one-dimensional models, where the propagation is easy to describe (for example, by means of the transfer matrix method [10] or Green function [11]). In higher dimensional models the localization requires stronger stochastic inhomogeneity of the periodic medium [12].

The paper of Anderson dealt with the diffusive transport model. In later papers there were considered other processes of transport or heat conduction [13]. There were applied more advanced and rigorous mathematical tools for the analysis of the phenomenon. In crystal lattices it was also observed the localization of vibrations of the atoms [14]. Very similar but more illustrative is the example of coupled pendula [15], [16]. In this model it is assumed that infinitely many pendula of random, independent natural frequencies of oscillations are arranged on a chain. The neighbours in the chain are coupled by springs so that it is possible propagation of vibrations of the pendula. However, in such a structure due to the randomness of natural vibration frequencies, the vibrations are localized.

The examples of more complicated models where the localization occurs are the domino billiard [17] or the waveguide with random boundaries [18].

The localization phenomenon was also observed in the models of periodic engineering structures with random disturbances of the periodicity. The examples can be mono-coupled engineering structures like shear buildings, long continuous beam on simple supports [19] or beam on hinge supports with additional torsional spring at each support [20].

A very wide class of physical processes is connected with the wave propagation in stochastically perturbed periodic media. Also in such models the localization phenomenon was observed, both in idealized stratified media and scalar waves [21] and in models of more complicated processes, like localization of hydrodynamical shallow surface waves due to scattering with a random bottom [22] or localization of electromagnetic waves in a plasma with a fluctuating density [23].

In this paper we wish to present, how the random stratification of the medium generates localization in stationary (harmonic waves) and non-stationary (wave pulses) cases.

3. WAVES IN RANDOMLY STRATIFIED MEDIUM AND THE LOCALIZATION

Consider planar harmonic wave propagating perpendicularly to the interfaces of strata in stratified medium. In such a case the problem can be considered as one-dimensional and conveniently described with application of the transfer matrix formalism (see [24]). This means that transition of the wave through each layer can be completely described by some matrix, called transfer matrix, with elements depending on the character of the wave (elastic, electromagnetic, e.t.c.), properties (material parameters) of the layer and its thickness. Transition through several neighbouring layers is then described by the product of corresponding transfer matrices.

Assume that the space where the wave propagates is homogeneous for x<0; for x>0 it is periodically stratified. The possibility of harmonic wave propagation in such a medium (that is for certain fixed material parameters of the materials of the strata) depends on the wave frequency (see [25]). For some frequencies wave propagation is impossible (the frequency is within stop bands), for other frequencies it is possible (the frequency is within pass bands). When the propagation is possible, the wave field $u(x)$ in the periodically stratified half-space is so-called Floquet (or Bloch) wave. It has the following form :

$$\begin{aligned} u(x) &= v(x)e^{i\lambda x} \\ v(x) &= v(x+a) \end{aligned}, \tag{1}$$

where *v(x)* is a periodic function with period *a*, corresponding to the thickness of the periodic strata in the stratified medium and λ is the Floquet wave number, calculated from the eigenproblem on the individual periodic strata (usually with the use of the matrix being the transition matrix through the strata).

Different situation appears when the stratification of the medium is random. We assume now, that the transition matrices through the layers in the stratified medium are independent and identically distributed random matrices. In such a case generation of the Floquet type wave is impossible. To study the wave process let us consider at the beginning the slab built of N layers of finite thickness, each with the transition matrix $\mathbf{T}(j,\gamma)$, $j=1,2,...,N$, surrounded by the homogeneous material. If we assume some incident harmonic wave coming from minus infinity, then in the homogeneous environment of the stratified slab we obtain the reflected wave and the transmitted wave. The equation for their amplitudes can be written with the use of transfer matrices in the following way [26]:

$$\begin{vmatrix} A \\ B \end{vmatrix} = \mathbf{M}^{-1} \prod_{j=1}^{N} \mathbf{T}(j,\gamma)\ \mathbf{M}_{fin} \begin{vmatrix} C \\ 0 \end{vmatrix}, \tag{2}$$

where A is the amplitude of the incident wave and B and C are the amplitudes of the reflected and transmitted waves and γ is a random element. In formula (2) matrix $\mathbf{M}^{-1}$ describes the reflection and transmission of the waves at the interface of the environment and the first layer of the slab, matrix $\mathbf{M}_{fin}$ describes the transmission of the wave from the last layer of the slab to the environment and the product of the transition matrices $\prod_{j=1}^{N} \mathbf{T}(j,\gamma)$ is responsible for the effect of the randomly stratified medium on the wave. In each layer of the slab the left-going and the right-going harmonic waves are generated. To study the wave propagation in the randomly stratified half-space we must take the limit $N \to \infty$ in equation (2). In the analysis of this problem the following theorem of Furstenberg [8] can be useful: *under certain* *conditions* (satisfied by the transition matrices in our wave propagation problem) *the product of the random* $m \times m$ *matrices* $\mathbf{T}(j,\gamma)$ *satisfies*:

$$N^{-1} \log \left\| \prod_{j=1}^{N} \mathbf{T}(j,\gamma)\mathbf{u} \right\| \xrightarrow[N\to\infty]{} \alpha \,, \tag{3}$$

where $\| \, . \, \|$ *is the norm in* R^m , α *is some positive constant and* $\mathbf{u}$ *is an arbitrary vector in* R^m. (The number α is the maximal Lyapunov exponent of the product of the matrices). Applying the formula (3) to our wave equation (2) we can observe, that if the number of layers in the medium (and, what it follows, the thickness of the randomly stratified slab) tends to infinity, then the amplitudes of the incident and reflected waves can remain bounded only if the amplitude of the transmitted wave tends to *0* exponentially; in the limit the amplitude *C* should vanish. This means that the randomly stratified half-space totally reflects the harmonic wave - in the stationary case the energy coming to the surface of the random medium is equal to the reflected energy. However, the reflection procedure takes place not only on the interface of the homogeneous half-space and the first strata, but also on the interfaces of the strata within the stratified half-space; the intensity of the reflection decreases along the distance from the surface of the stratified half-space. In such a case we have the localization of the wave, interpreted in two ways. Firstly, the wave cannot propagate in the randomly stratified medium and is stopped (localized) by it. Secondly, in the area next to the interface of the media permanently exist the waves (left- and right-going) and, what it follows, the localized energy of the waves (see Fig.1). Looking for the analogy with the Anderson model of the electron transport in the crystal lattice we can say, that this situation corresponds to the case of all states being localized (no transport - no propagation).

The localization phenomenon can be also observed in the process of wave pulse transition through the elements of structures (bars) of finite length (see [27]). In contradiction to the case of the randomly stratified half-space, where the incident harmonic wave is completely reflected, the wave pulse partially reflects from the bar and partially transmits through it. The problem is non-stationary and, what it follows, more complicated to describe than stationary harmonic one. However, using the Fourier transform method we can solve it.

Consider the bar built of several segment, each with the (finite) length and constant material parameters being random variables (see Fig.2), surrounded by a homogeneous medium. In the segments the longitudinal wave pulse can be described by the following partial differential equation:

$$\frac{\partial}{\partial \tau} \begin{vmatrix} f \\ v \end{vmatrix} = \begin{vmatrix} 0 & Z \\ \frac{1}{Z} & 0 \end{vmatrix} \frac{\partial}{\partial t} \begin{vmatrix} f \\ v \end{vmatrix}, \tag{4}$$

where f is the force, v is the particle's velocity, Z is the characteristic impedance of the segment of the bar and the variables t and τ represent time and the travel time through the bar (see [28]). Fourier transformation of equation (4) with respect to variable t, elimination of the velocity v and application of the transition matrix method to the whole segmented bar let us to write the wave equation in the matrix form analogous to equation (2):

$$\begin{vmatrix} F_{inc}(\omega) \\ F_{ref}(\omega) \end{vmatrix} = \mathbf{M}^{-1}(\omega, Z_0) \prod_{j=1}^{N} \mathbf{T}(\omega, Z_j, \tau_j) \; \mathbf{M}_{fin}(\omega, Z_0) \begin{vmatrix} F_{tr}(\omega) \\ 0 \end{vmatrix}, \tag{5}$$

where F_{inc}, F_{ref} and F_{tr} are the Fourier transforms of, respectively, the incident, reflected and transmitted pulses, ω is the Fourier transform spectral parameter and $\mathbf{T}(\omega, Z_j, \tau_j)$ is the transition matrix through j-th segment of the bar, depending on the parameter ω, characteristic impedance Z_j and the wave travel time through the segment τ_j (see [27]):

$$\mathbf{T}(\omega, Z_j, \tau_j) = \begin{vmatrix} \cos\omega\tau_j & Z_j \sin\omega\tau_j \\ -\frac{1}{Z_j}\sin\omega\tau_j & \cos\omega\tau_j \end{vmatrix} . \tag{6}$$

Matrices $\mathbf{M}^{-1}(\omega, Z_0)$ and $\mathbf{M}_{fin}(\omega, Z_0)$ describe the transition of the wave pulse from the environment to the bar and from the bar to the environment. As it is seen from the formula (5), the product of random transition matrices describe the complicated reflection and transmission process that takes place when some incident (force) pulse reaches the front of the bar. As it is known, the wave pulse at each interface of segments partially reflects and partially transmits (see Fig. 2); also the reflected pulses reflect and transmit, so the final picture of reflected and transmitted pulses becomes very complicated. From the trace of the pulses we can observe that the initial pulse, crossing the interfaces of the segments, reflects so many times that it practically disappears. On the other hand some later pulse is being formatted due to summing up a number of multiple reflected pulses. Finally, when the number of segments in the bar is going to infinity (but with limited total length of the bar) some new transmitted pulse is created. This transmitted pulse is similar to some pulse that would go through a homogeneous bar, but coming later. This procedure of infinitely many reflections and transmissions of the pulses at the interfaces of the segments generates the effect of localization of the wave - analogous to the Anderson localization in the case of a moderate inhomogeneity of the crystal lattice: the transmission is saved but it is slower than in a homogeneous case. Moreover, inside the randomly segmented bar more wave energy is concentrated during the process than in some analogous homogeneous bar.

Using the formulae (5) and (6) we can describe the localization phenomenon quantitatively. As an example of application consider the bar built periodically of N couples of segments, where in the couple one segment is made of material 1 and another - of material 2. The impedances and lengths (or, what is equivalent, the travel times) of the repeating elements are independent, identically distributed (i.i.d.) random variables. This makes that the transition matrices through the couples of segments are also i.i.d. random matrices. Consider the case where the number of segments in the bar N tends to infinity but the total transition time through the bar (and through each of the materials in the bar), remains constant. Then, applying the limit theorem by Berger [29] we obtain from (5) the limiting formula which proves to be identical to the equation for the wave amplitudes of some homogeneous bar with effective parameters (impedance and travel time). For example, the expression for the effective travel time τ_{eff} is

$$\tau_{eff} = \sqrt{\left[E\{Z_1(\gamma)\tau_1(\gamma)\} + E\{Z_2(\gamma)\tau_2(\gamma)\}\right]\left[E\{\tau_1/Z_1\} + E\{\tau_2/Z_2\}\right]} \ , \tag{7}$$

where $Z_i(\gamma)$ and $\tau_i(\gamma)$, $i = 1,2,$ are, respectively, the characteristic impedance of material 1 and 2 and total travel time through the segments made of material 1 and 2.

Consider finally the numerical example of the bar built of two materials: one of them with impedance $Z_1 = 2$ and another with impedance $Z_2 = 4$, surrounded with an environment of impedance $Z_0 = 1$. We assume that both materials have such a share within the bar that the average travel time through each of the materials is equal to 1 (the sum of the expected values of the travel times through all the segments is 1). We assume that the force wave pulse of unite duration and unite amplitude reaches the bar at time 0. Figure 3 shows the amplitude of the pulse transmitted through the bar built of 150 couples of segments with random lengths. It can be observed the concentration of pulses creating the big pulse analogous to one obtained for a homogeneous bar. It is also visible that such a pulse needs more time to pass the bar than it could be deduced from simple addition of the travel times through the component materials (such a sum would be equal to 2). Indeed, applying formula (7) we obtain that for our material parameters the effective travel time through the randomly segmented bar is equal to 2.1213. In this way we have been able to express the localization effect not only qualitatively (finding the great amount of reflections and transmissions of the pulses), but also quantitatively, calculating the increase of the effective travel time through the randomly segmented bar.

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REFERENCES

1. P.W.Anderson, Absence of diffusion in certain random lattices, *Physical Review*, vol.109, no.5, pp.1492-1505, (1958).
2. F.Martinelli, E.Scoppola, Introduction to the mathematical theory of Anderson localization, *Rivista del Nuvo Cimento*, vol.10, no.10, pp. 1-90, (1987).
3. M.M.Benderskii, L.A.Pastur, On the spectrum of the one dimensional Schrödinger equations with a random potential, *Math USSR Sb*. vol.11, p.245, (1970).
4. R.Lang, Spectral Theory of Random Schrödinger Operators. A.Genetic Introduction. *Lecture Notes in Mathematics* no. 1498, Springer Verlag, Heidelberg (1991).
5. I.Ya.Gol'dshtein, S.A.Molchanov, L.A.Pastur, A pure point spectrum of the stochastic one-dimensional Schrödinger operator, *Functional Analysis and its Applications*, vol.11, no.1, pp.1-8, (1977).
6. L.A.Pastur, Spectral theory of random self-adjoint operators, *Journal of Soviet Mathematics,* vol.46, no.4, pp.1977-2021, (1989).
7. S.Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, in: K.Itô, (Ed.), *Stochastic Analysis*, Proc. Taniguchi Symp, Katata 1982, pp.225-247, North-Holland, Amsterdam (1984).
8. H.Furstenberg, Non-commuting random products, *Trans. AMS*, vol.108, pp.377-428, (1963).
9. R.Z.Khas'minskii, Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, *Theory of Probability and Its Applications*, vol.12, pp.144-147, (1967).
10. Z.Kotulski, Elastic waves in randomly stratified medium. Part I, Analytical results, *Acta Mechanica*, vol. 83, pp.61-75, (1990), Part II, Numerical calculations, *Acta Mechanica*, vol.92, pp.143-163, (1992).
11. J.Frohlich, T.Spencer, Absence of diffusion in the Anderson tight binding model for large disorder for low energy, *Commun. Math. Phys*., vol.88, p.151, (1983).
12. F.Delyon, Y.Levy, B.Souillard, Anderson localization for multi-dimensional systems at large disorder or low energy, *Commun. Math. Phys.*, vol.100, pp.463-470, (1985).
13. K.Ishii, Localization of eigenstates and transport phenomena in the one-dimensional disordered system, *Suppl of Progress of Theoretical Physics*, no.53, pp.77-138, (1973).
14. F.Delyon, H.Kunz, B.Souillard, One-dimensional wave equation in disordered media, *J.Phys. A*., vol.16, p.25, (1983).
15. C.H.Hodges, Confinement of vibration by structural irregularity, *Journal of Sound and Vibration*, vol.82, no.3, pp.411-424, (1982).
16. C.H.Hodges, J.Woodhouse, Theories of noise and vibration transmission in complex structures, *Rep.Prog.Phys*, vol.49, pp.107-170, (1986).
17. T.Dittrich, E.Doros, U.Smilansky, Classical diffusion, Anderson localization and spectral statistics in billiard chains, *preprint*.

18. F.Martinelli, The wave equation in random domains: localization of the normal modes in the small frequency region, *Ann.Inst.Henri Poincare*, vol.43, p.22, (1985).
19. S.T.Ariaratnam, Wei-Chau Xie, On the localization phenomenon in randomly disordered engineering structures, in: N.Bellomo, F.Casciati (Eds.), *Non-linear Stochastic Mechanics,* IUTAM Symposium Turin, pp. 13-24, Springer-Verlag, Heidelberg (1992).
20. G.Q.Cai, Y.K.Lin, Localization of wave propagation in disordered periodic structures, *AIAA Journal*, vol.29, no.3, pp.450-456, (1991).
21. S.A.Gredeskul, V.D.Freylikher, Localization and propagation of waves in randomly stratified media, *Uspiekhi Fizicheskikh Nauk*, vol.160, no.2, pp.239-262, (1990).
22. P.Devillard, F.Dunlop, B.Souillard, Localization of gravity waves on a channel with a random bottom, *J.Fluid Mechanics*, vol.186, pp. 521-538, (1988).
23. D.Escande, B.Souillard, Localization of waves in a fluctuating plasma, *Phys.Rev.Lett.,* vol.52, p.1296, (1984).
24. Z.Kotulski, Wave propagation in a randomly stratified medium, *Journal of Sound and Vibration*, vol.128, no.2, pp.195-208, (1989).
25. L.Brillouin, *Wave Propagation in Periodic Structures: Electric Filters and Crystal Lattices*, McGraw-Hill, New York (1946).
26. Z.Kotulski, Wave propagation in randomly stratified media and the law of large numbers, *Journal of Sound and Vibration*, vol.158, no.1, pp.93-104, (1992).
27. Z.Kotulski, On the effective reflection properties of the randomly segmented bar, *European Journal of Mechanics, A/Solids.* (to appear, 1994).
28. B.Lundberg, R.Gupta, L.-E.Anderson, Optimum transmission of elastic waves through joints, *Wave Motion*, vol.1, pp.193-200, (1979).
29. M.A.Berger, Central limit theorem for product of random matrices, *Trans. AMS*, vol. 285, pp.777-803, (1984).

Figure captions

Fig.1 The amplitudes of the right and left going harmonic waves decrease exponentially along the depth of the randomly stratified half-space.

Fig.2 Evolution in time of the front of the wave pulse in the randomly segmented elastic bar.

Fig.3 The sample path of the amplitude of the reflected pulse for the bar built of 150 couples of segments with random lengths.